CS134 Lecture 21:
Graphical Recursion

Announcements & Logistics

- **Lab 7** today and tomorrow: focuses on **recursion**
  - Please write/print the **pre lab** before you come to lab
  - Partner lab: you and your partner have to attend the same lab section
  - We will be collecting it at the start of lab.
  - Prelab is an ***individual*** assignment
    - You may discuss with your partner after submitting it
- **HW 6** due @ 10 pm
  - We made a mistake on one question — Glow is now fixed. The question no longer counts against your quiz score.

Do You Have Any Questions?

Maximum coverage at **3.27 pm** at Williams College

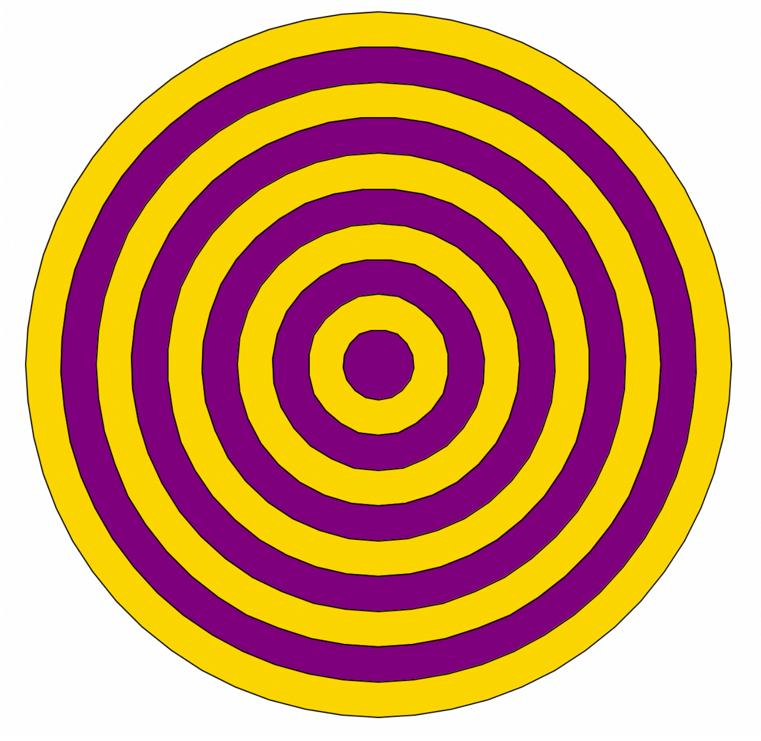
For the 2.30pm lab folks: we can walk out to watch for a few mins, so bring your eclipse viewing glasses!

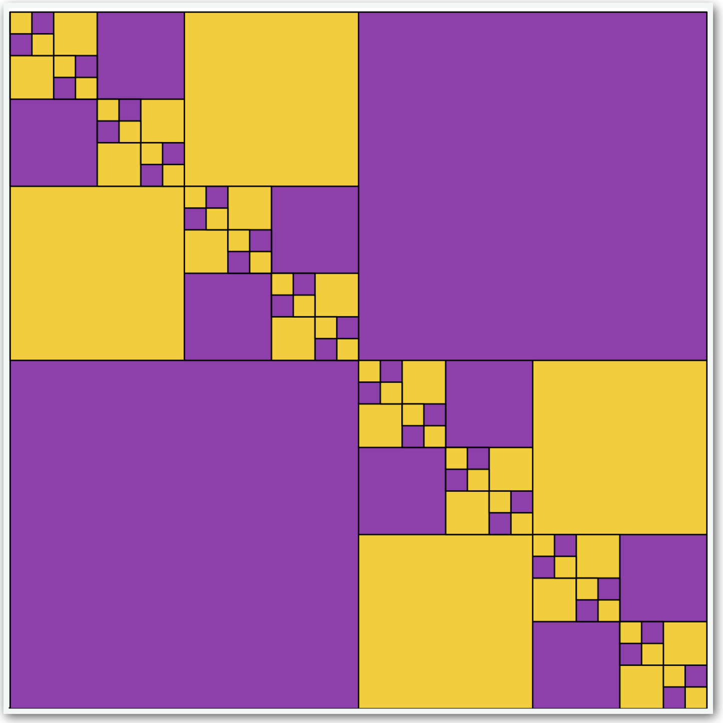
Last Time: Recursive Approach to Problem Solving

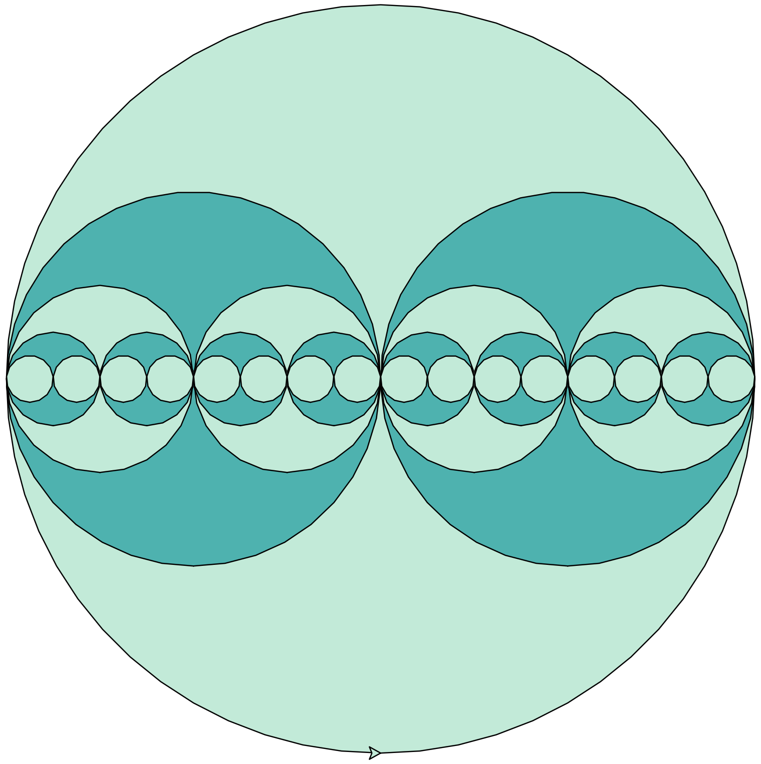
- A recursive function is a function **that calls itself**
- A recursive approach to problem solving has two main parts:
  - **Base case(s).** When the problem is **so small**, we solve it directly, without having to reduce it any further
  - **Recursive step.** Does the following things:
    - Performs an action that contributes to the solution
    - **Reduces** the problem to a smaller version of the same problem, and calls the function on this **smaller subproblem**
- The recursive step is a form of "wishful thinking" (also called the inductive hypothesis)

Today's Plan

- Introduction to Turtle
- Graphical recursion examples
- Understanding function **invariance** and why it matters when doing recursion







The Turtle Module

- Turtle is a **graphics module** first introduced in the 1960s by computer scientists Seymour Papert, Wally Feurzig, and Cynthia Solomon.
- It uses a programmable cursor — fondly referred to as the “turtle” — to draw on a Cartesian plane (x and y axis.)

Turtle In Python

- `turtle` is available as a built-in module in Python. See the Python turtle module API for details.
- Basic turtle commands:

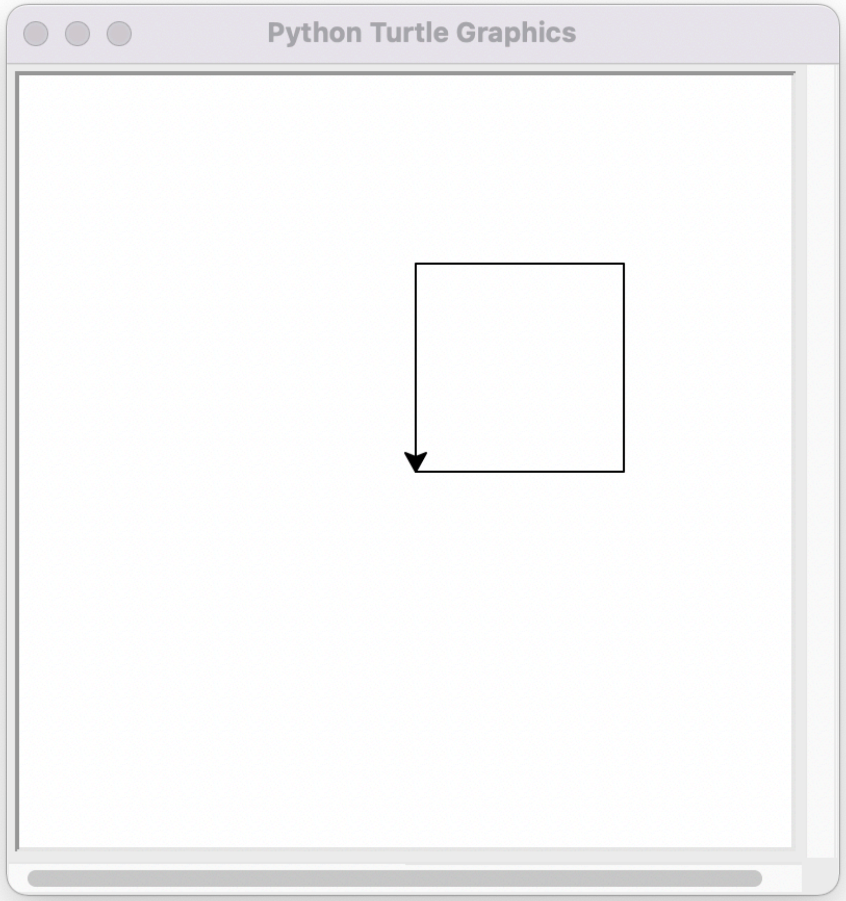
Use `from turtle import *` to use these commands

`fd(dist)`	turtle moves forward by dist
`bk(dist)`	turtle moves backward by dist
`lt(angle)`	turtle turns left angle degrees
`rt(angle)`	turtle turns right angle degrees
`up()`	(pen up) turtle raises pen in belly
`down()`	(pen down) turtle lowers pen from belly
`shape(shp)`	sets the turtle's shape to shp
`speed(spd)`	sets the turtle's speed 1-10 (slow-fast). 0 skips animation.
`home()`	turtle returns to (0,0) (center of screen)
`clear()`	delete turtle drawings; no change to turtle's state
`reset()`	delete turtle drawings; reset turtle's state
`setup(width, height)`	create a turtle window of given width and height

Basic Turtle Movement

- forward(dist) or fd(dist), left(angle) or lt(angle), right(angle) or rt(angle), backward(dist) or bk(dist)

```
# set up a 400x400 turtle window
setup(400, 400)
reset()

fd(100) # move the turtle forward 100 pixels

lt(90) # turn the turtle 90 degrees to the left

fd(100) # move forward another 100 pixels

# complete a square
lt(90)
fd(100)
lt(90)
fd(100)
done()
```

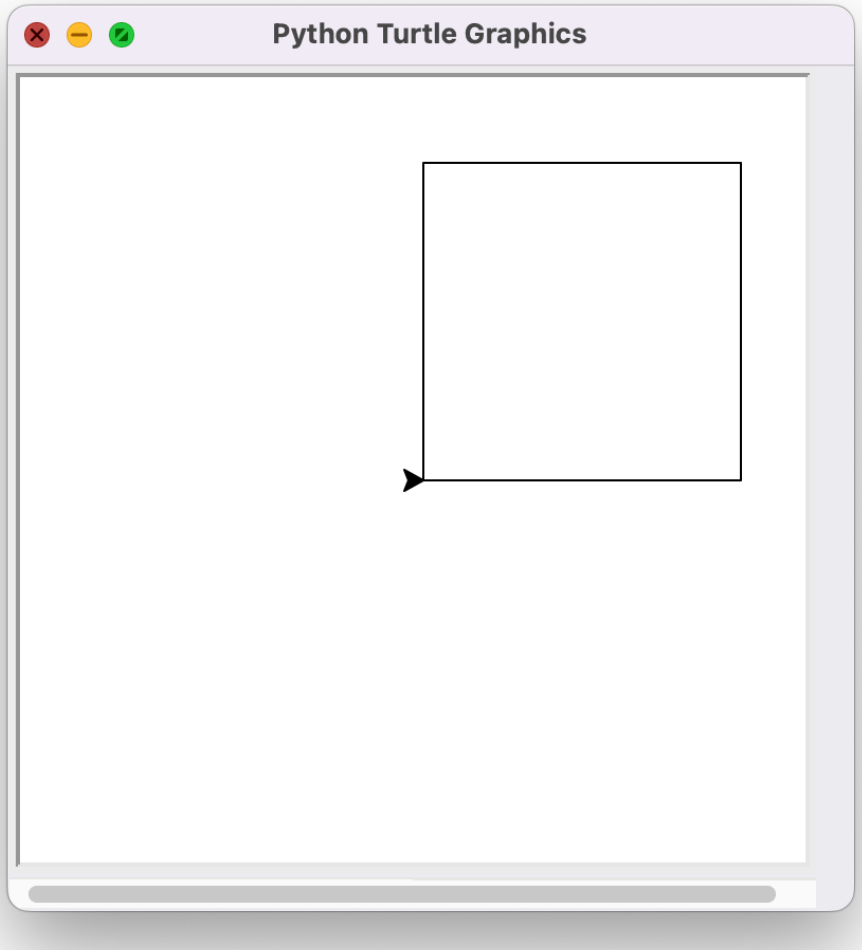


Drawing Basic Shapes With Turtle

- We can write functions that use turtle commands to draw shapes.
- For example, here’s a function that draws a square of the desired size

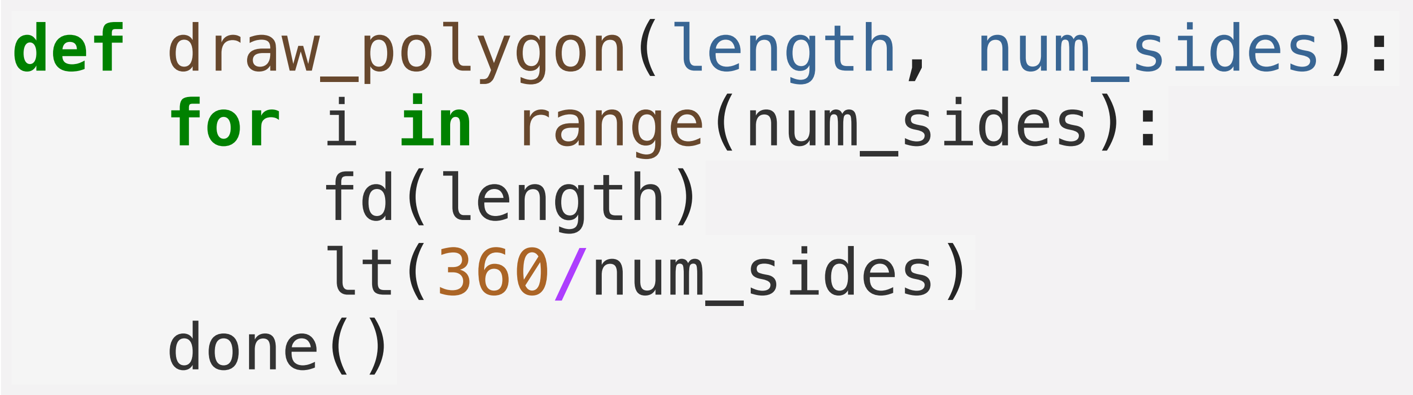
```
def draw_square(length):
    # a loop that runs 4 times
    # and draws each side of the square
    for i in range(4):
        fd(length)
        lt(90)
    done()
```

```
setup(400, 400)
reset()
draw_square(150)
```

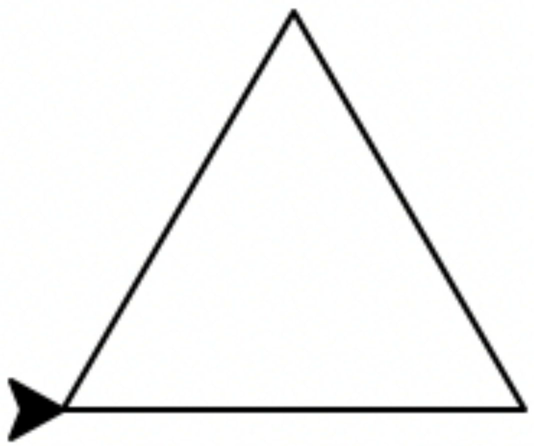


Drawing Basic Shapes With Turtle

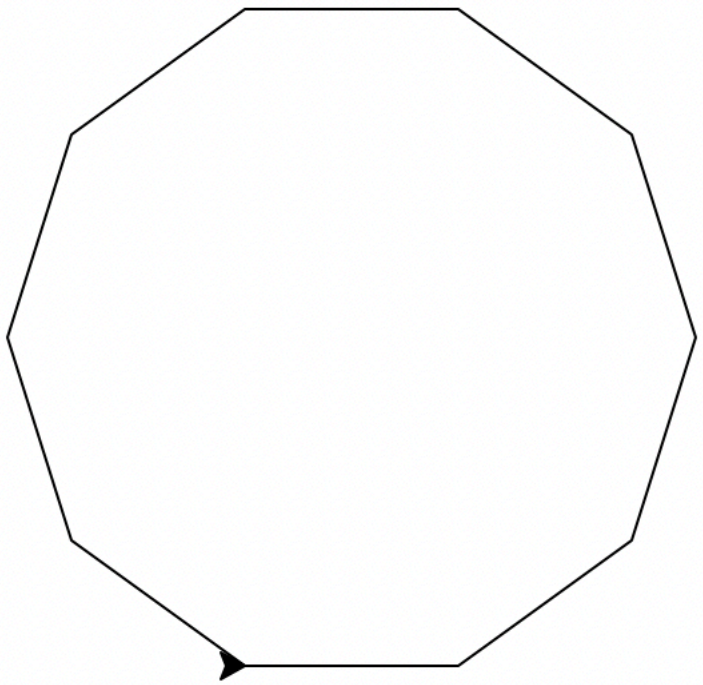
- How about drawing polygons?



```
def draw_polygon(length, num_sides):
    for i in range(num_sides):
        fd(length)
        lt(360/num_sides)
    done()
```



```
draw_polygon(80, 3)
```

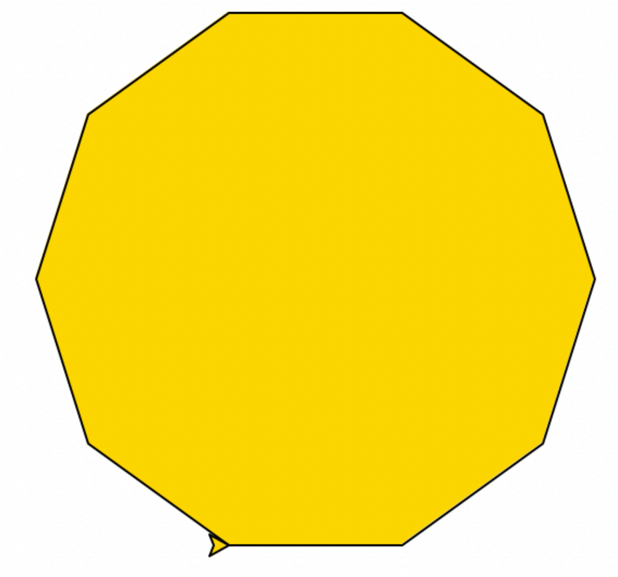


```
draw_polygon(80, 10)
```

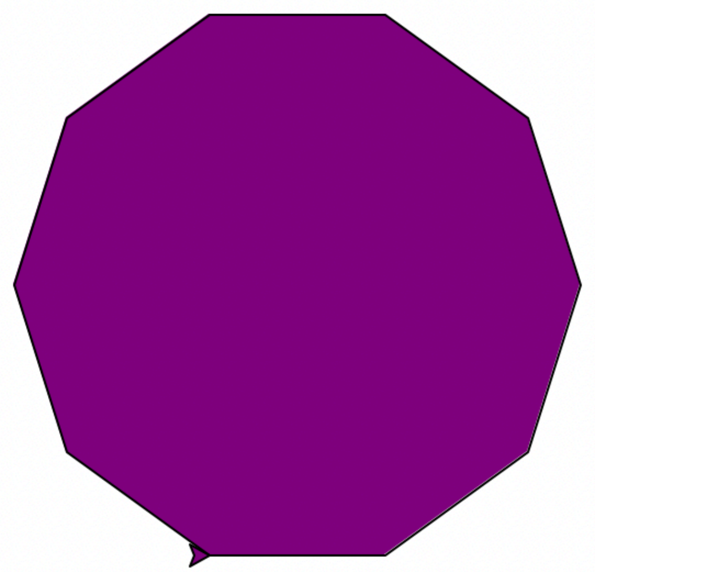
Adding Color!

- What if we wanted to add some color to our shapes?

```python
def draw_polygon_color(length, num_sides, color):
    # set the color we want to fill the shape with
    # color is a string
    fillcolor(color)

    begin_fill()
    for i in range(num_sides):
        fd(length)
        lt(360/num_sides)
    end_fill()
    done()
```



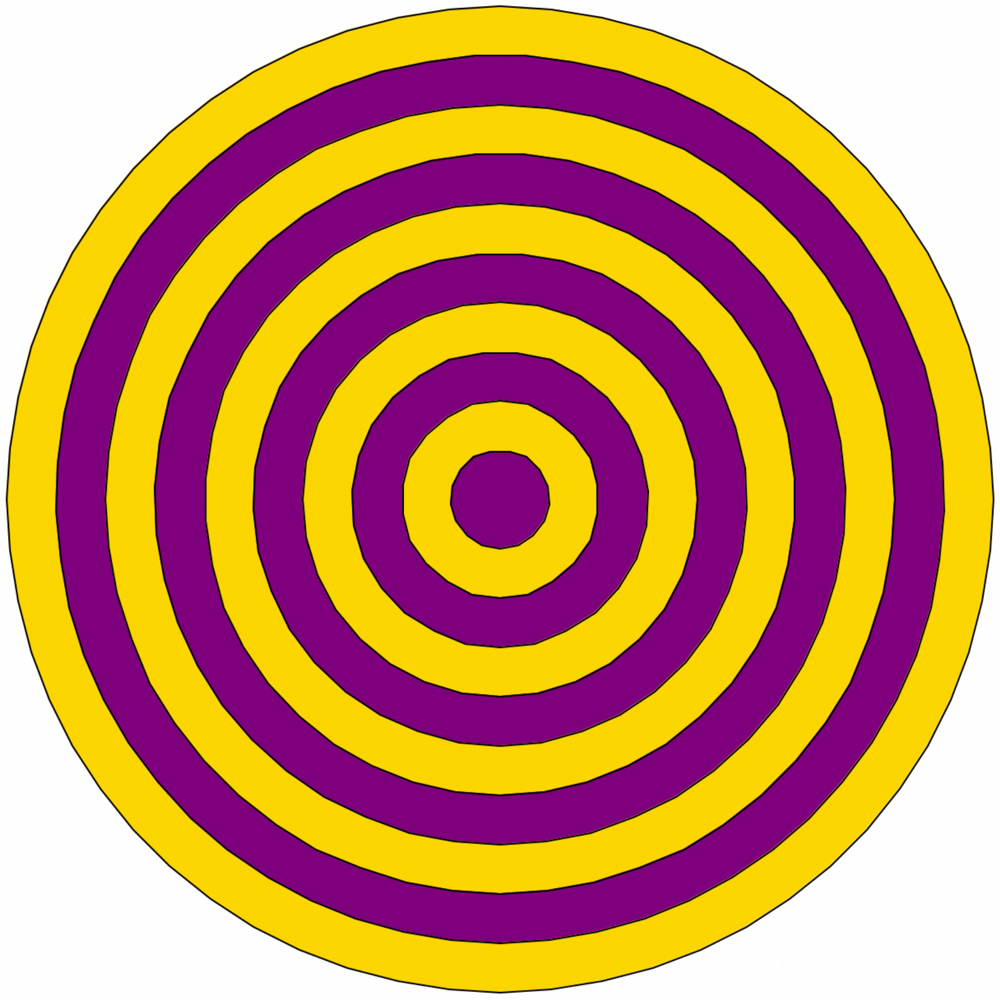
```python
draw_polygon_color(80, 10, "gold")
```



```python
draw_polygon_color(80, 10, "purple")
```

Recursive Figures With Turtle

- Let's explore how to draw pretty recursive pictures with Turtle
- We'll start with figures that only require recursive calls
- Below we have a set of concentric circles of alternating colors
- How is this recursive?



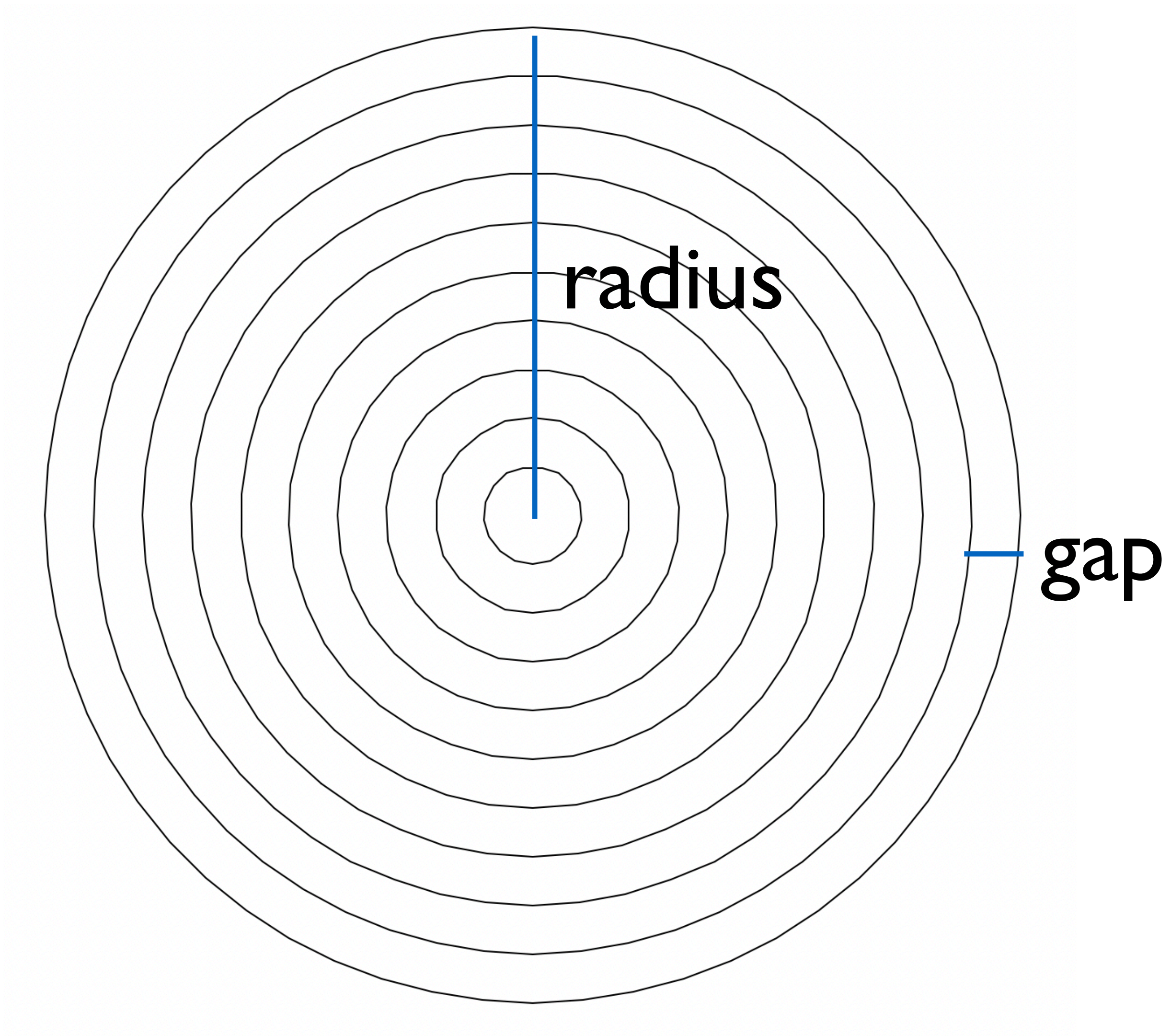
Example:
Concentric Circles

Concentric Circles

- Function definition

```
concentric_circles(radius, gap)
```

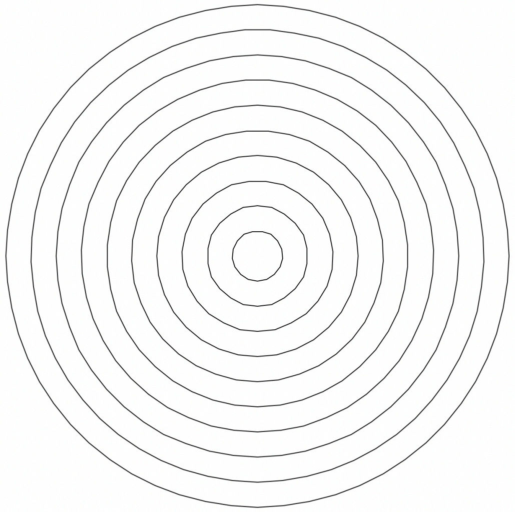
  - `radius`: radius of the outermost circle
  - `gap`: width of gap between circles



Concentric Circles With No Colors

- Let's first think about the circles without colors.
- **Base case**: radius of the circle is so small it's not worth drawing
- **Recursive step**:
  - Draw a single circle of radius `r`, increment total by 1
  - Recursively draw concentric circles starting with an outer circle of a slightly smaller radius `r-g` (where `g` is any positive number you want to shrink the radius by, or the "gap" between the circles)
- Let's also count the number of circles we draw to understand the process

Counting the number of circles isn't necessary for drawing pictures, but it does make debugging easier!



Concentric Circles

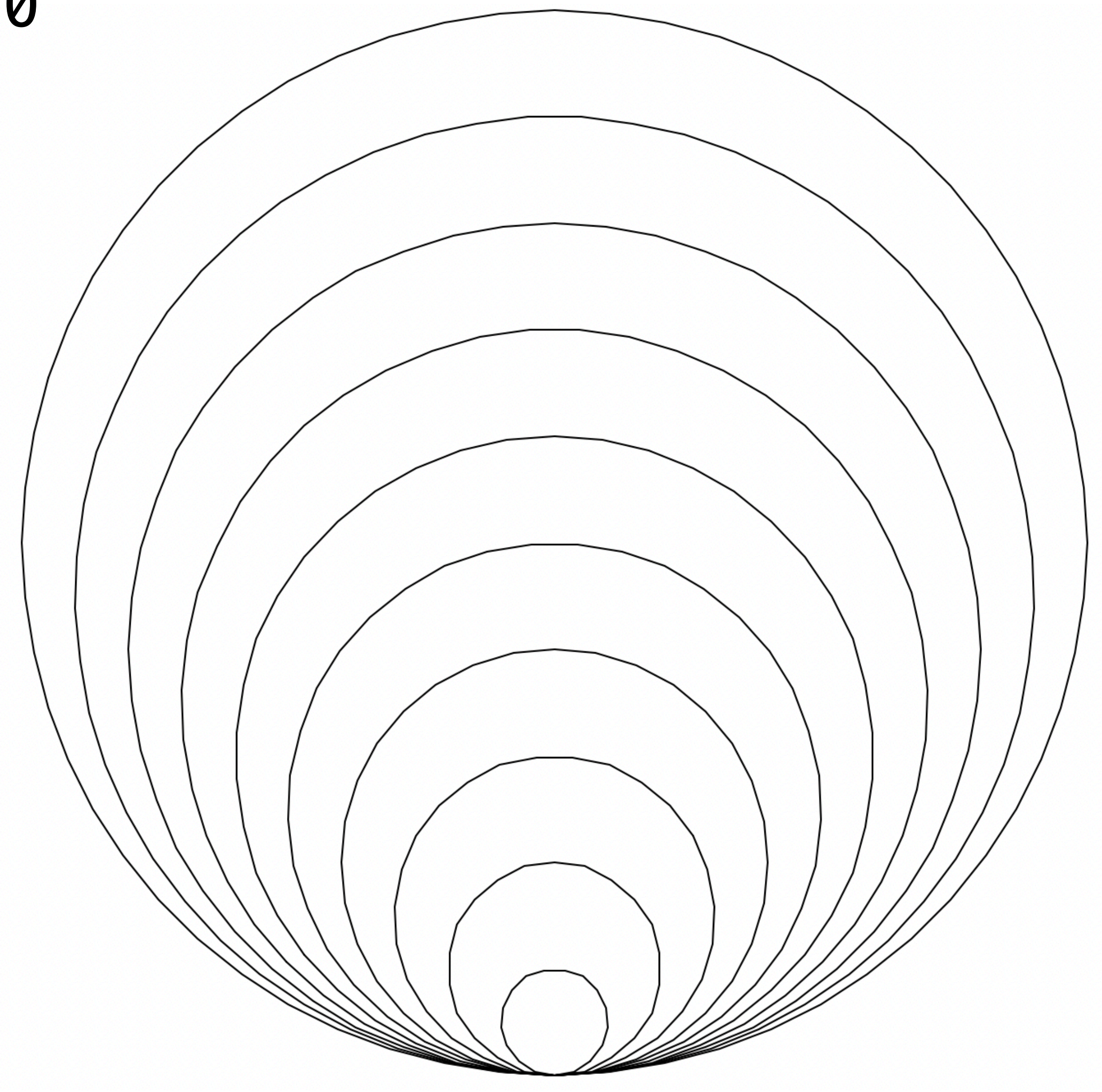
```
def concentric_circles(radius, gap):
    '''draw concentric circles and return # circles drawn'''
    # base case, don't draw anything, return 0
    if radius < gap:
        return 0
    else:
        # tell the turtle draw a circle
        circle(radius)

        # recursive function call; draw smaller circles
        num = concentric_circles(radius-gap, gap)

        # we drew one circle in this step, plus however many we
        # drew recursively, so return 1 + num
        return 1 + num
```

- Are we done?

Concentric Circles

```
print("Num Circles:", concentric_circles(300, 30))
```

```
Num Circles: 10
```

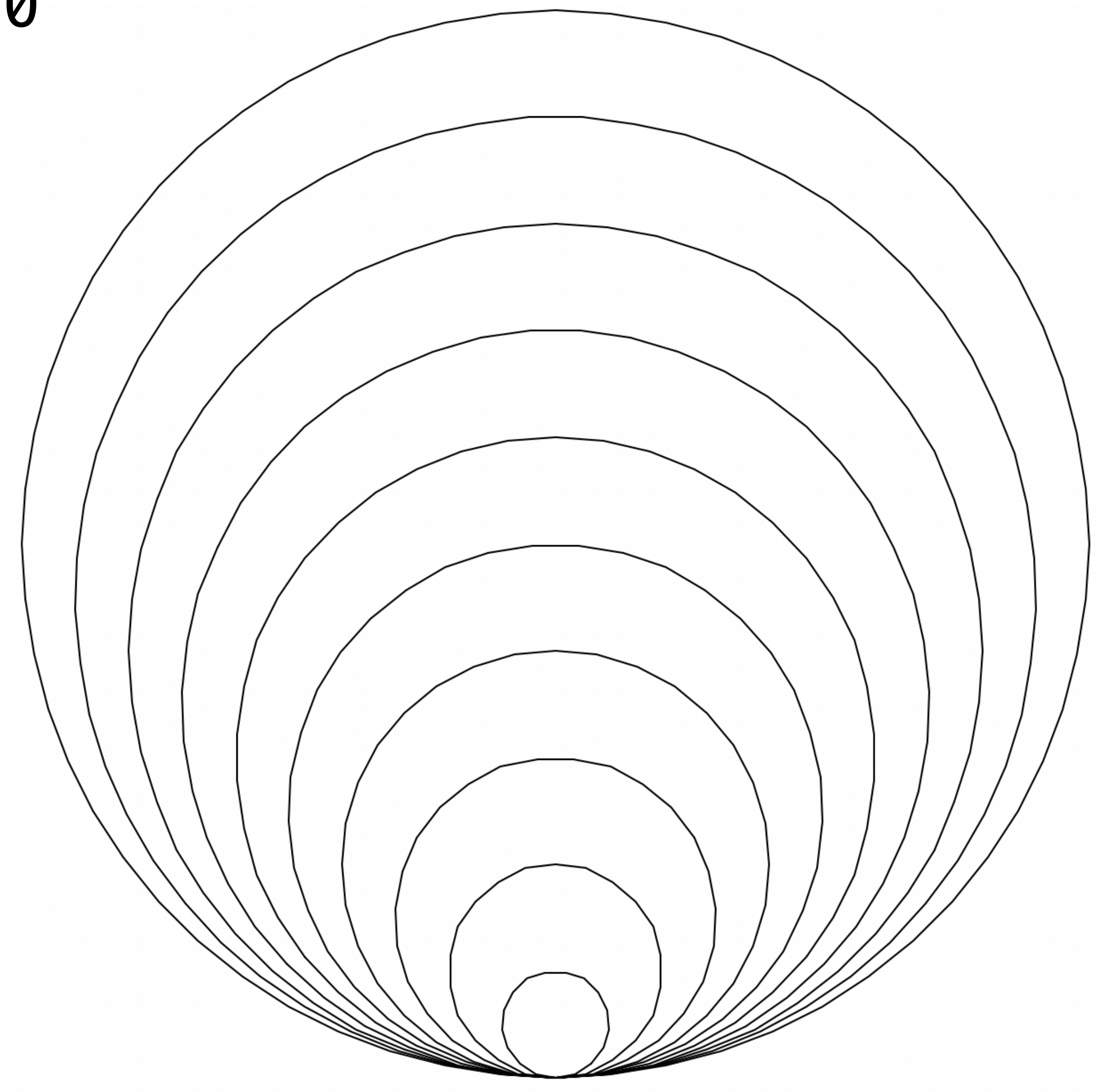


- Pretty picture, and almost there! But not quite right. What happened?

Concentric Circles

```
print("Num Circles:", concentric_circles(300, 30))
```

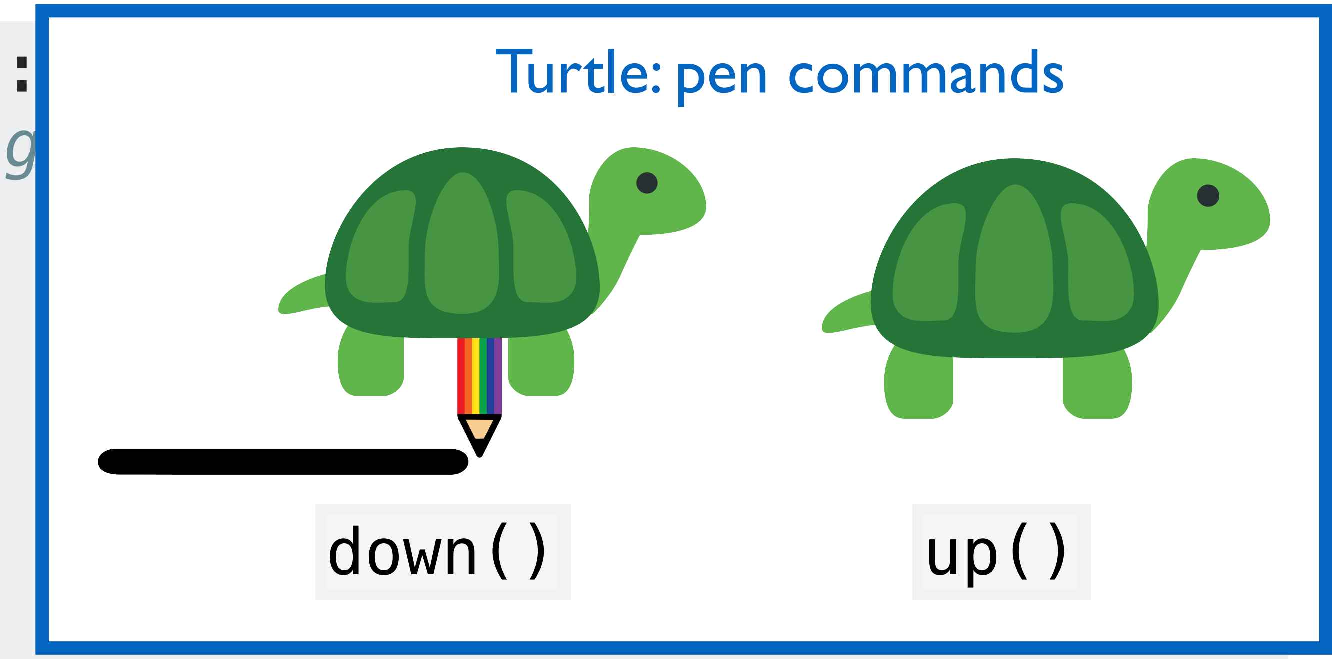
```
Num Circles: 10
```



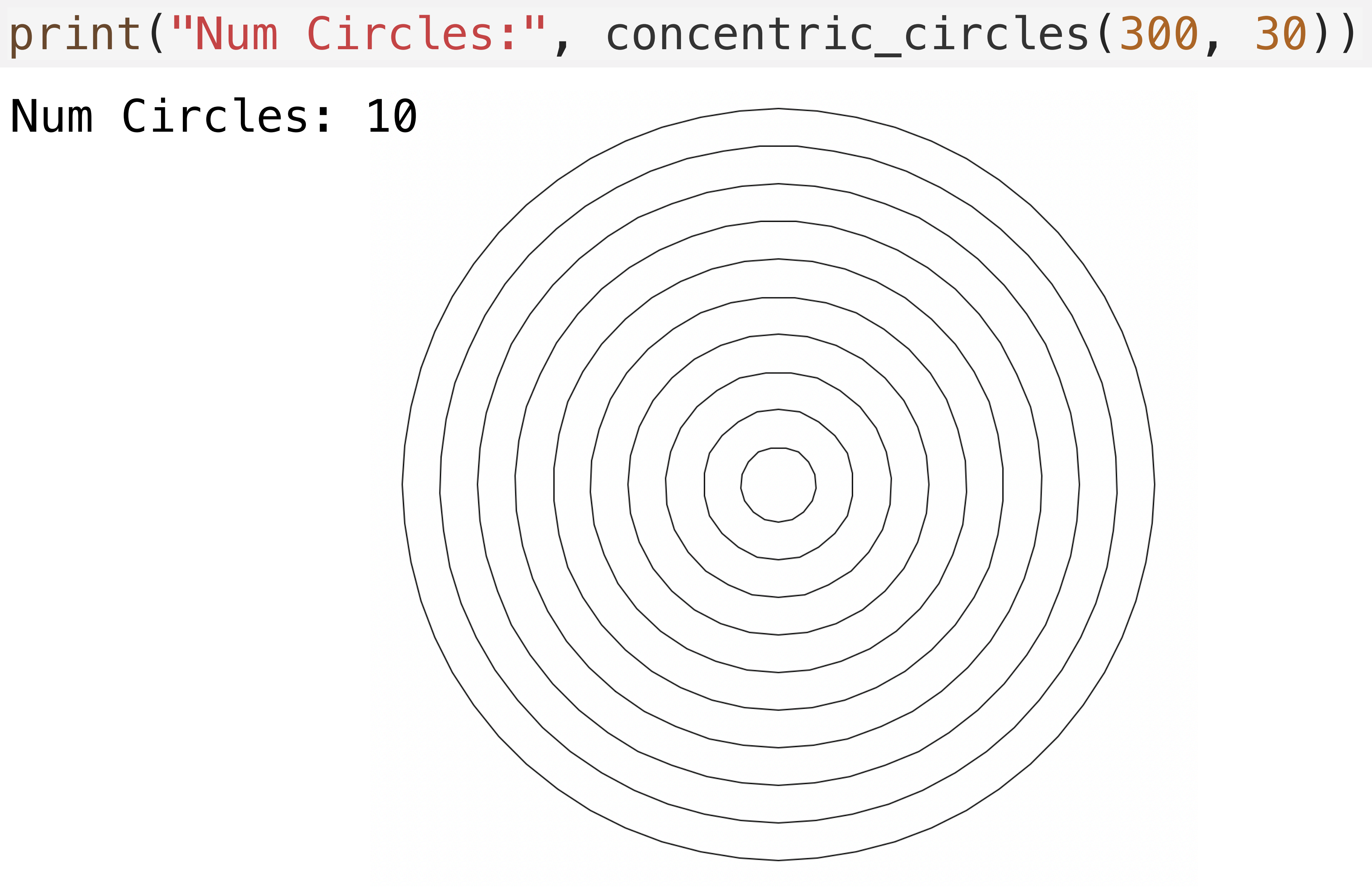
- We need to reposition the turtle after each recursive call.

Concentric Circles

```python
def concentric_circles(radius, gap):
    # base case, don't draw anything
    if radius < gap:
        return 0
    else:
        # pen down, draw circle
        down()
        circle(radius)

        # pen up, ensure the turtle doesn't draw while repositioning
        up()

        # reposition the turtle for the next circle
        lt(90)
        fd(gap)
        rt(90)

        # recursive function call; draw smaller circles
        num = concentric_circles(radius-gap, gap)

        # we drew one circle in this step, plus however many we
        # drew recursively, so return 1 + num
        return 1 + num
```


Turtle: pen commands

Concentric Circles



```
print("Num Circles:", concentric_circles(300, 30))
```

```
Num Circles: 10
```

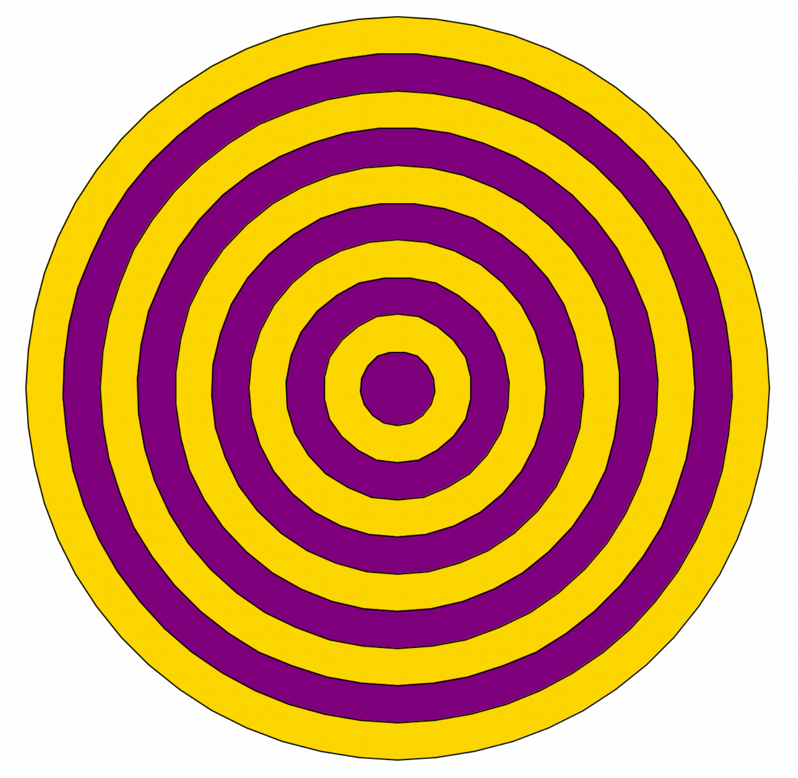
- Great! Now let's add some color.

Concentric Circles With Colors

- Function definition

```
concentric_circles(radius, gap, color_outer, color_inner)
```

- `radius`: radius of the outermost circle
- `gap`: width of the gap between circles
- `color_outer`: color of the outermost circle
- `color_inner`: color that alternates with color_outer



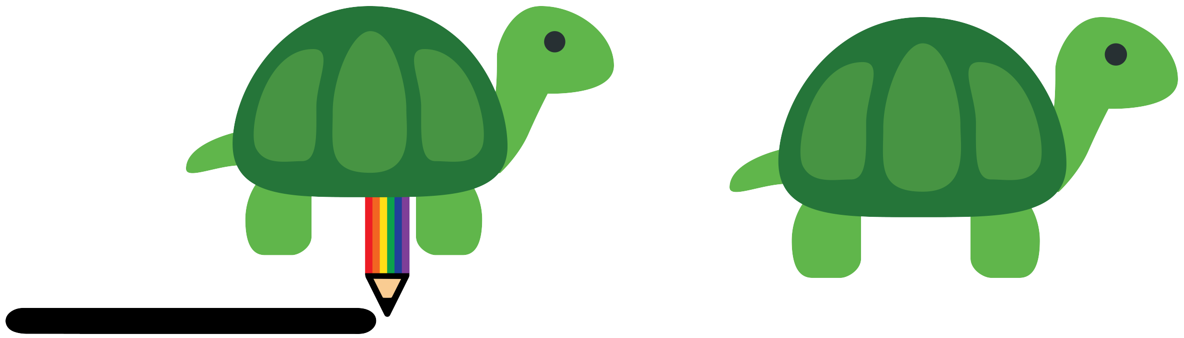
Concentric Circles: Adding Color

- Base case and recursive case stay the same
- How do we achieve the alternating colors?
- Just swap the order in the recursive call
  - `color_outer` becomes `color_inner` and vice versa
- Let's also write a helper function to draw a circle filled in with some color to clean up the recursive function itself

Helper Function

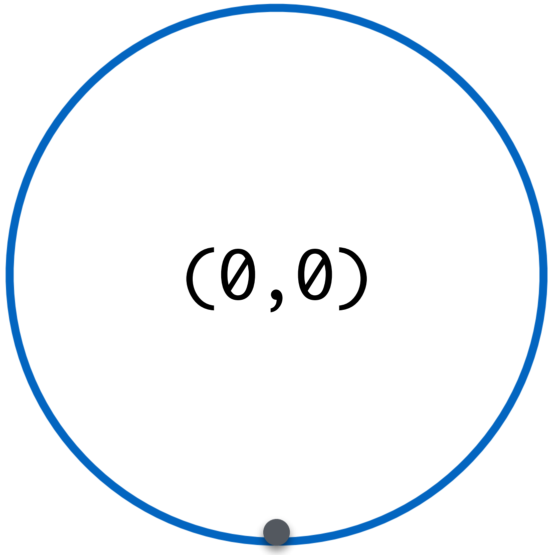
```python
def draw_disc(radius, color):
    """
    Draw circle of a given radius
    and fill it with color
    """

    # put the pen down
    down()

    # set the color
    fillcolor(color)

    # draw the circle
    begin_fill()
    circle(radius)
    end_fill()

    # put the pen up
    up()
```

Turtle: pen commands



`down()`

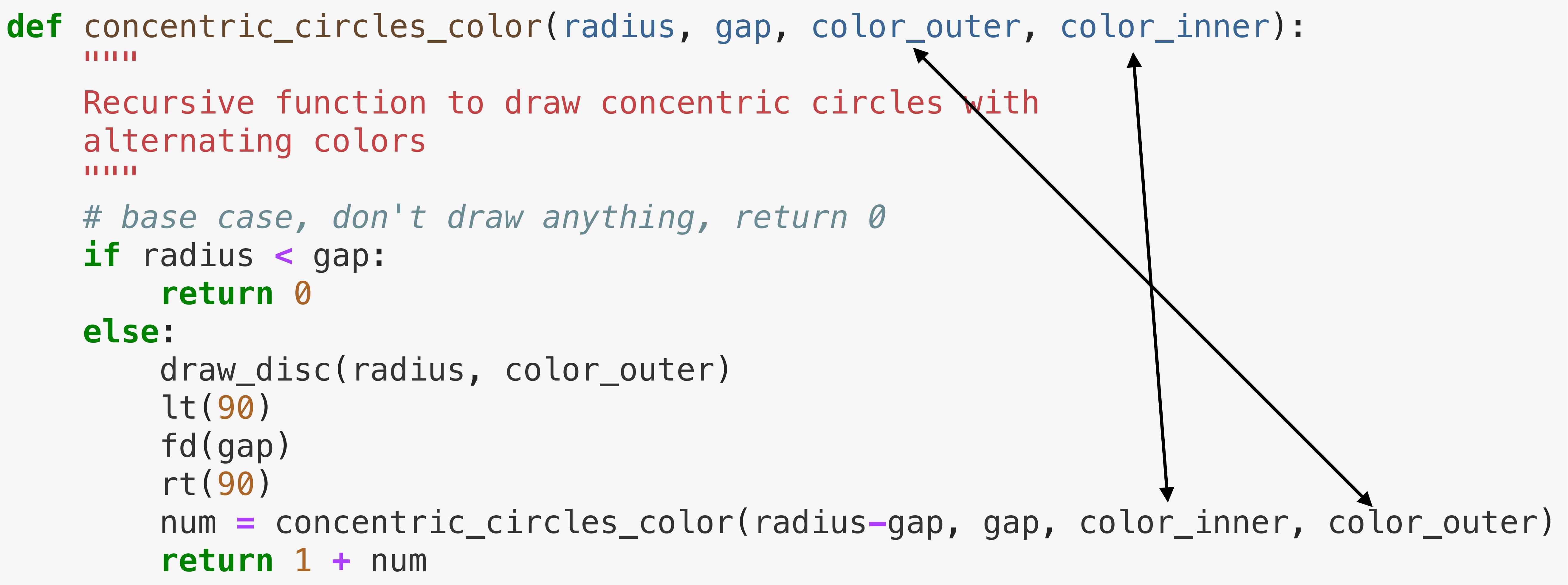
`up()`



Starting position of turtle

`(0, -radius)`

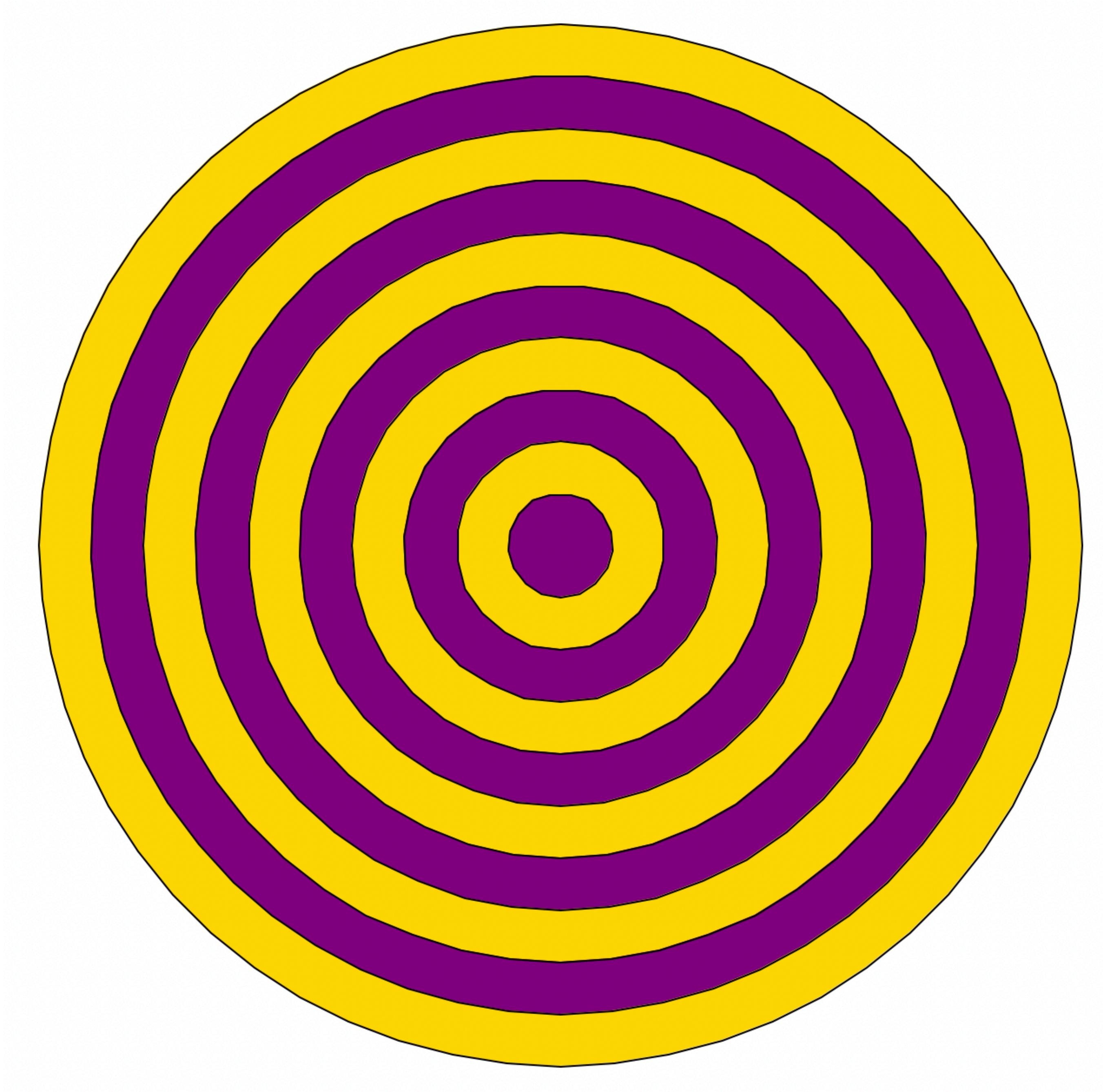
The Recursive Function



```
def concentric_circles_color(radius, gap, color_outer, color_inner):
    """
    Recursive function to draw concentric circles with
    alternating colors
    """
    # base case, don't draw anything, return 0
    if radius < gap:
        return 0
    else:
        draw_disc(radius, color_outer)
        lt(90)
        fd(gap)
        rt(90)
        num = concentric_circles_color(radius-gap, gap, color_inner, color_outer)
        return 1 + num
```

Concentric Circles

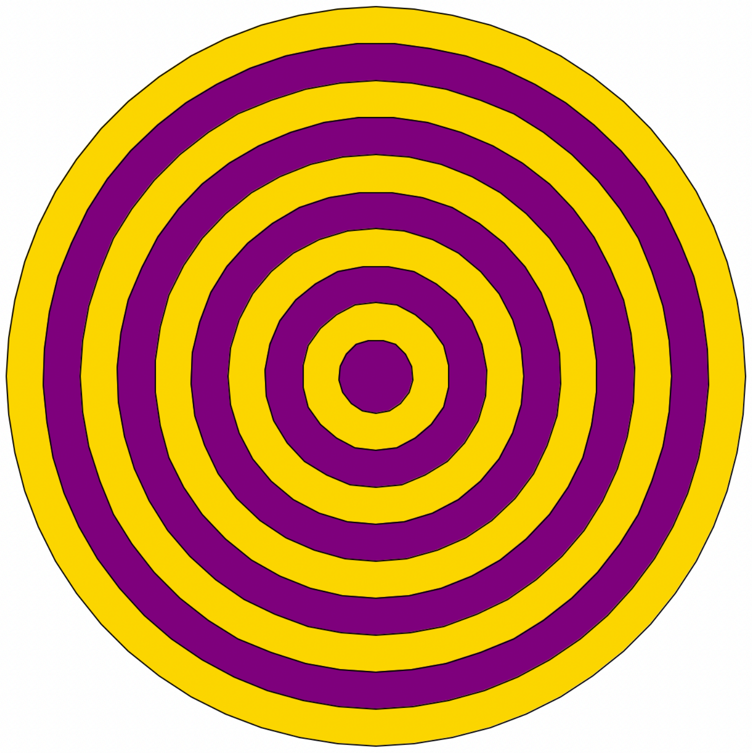
```
print("Num circles:", concentric_circles_color(300, 30, "gold", "purple"))
Num Circles: 10
```



Function Frame Model:

`concentric_circles`

```
def concentric_circles(radius, gap, color_outer, color_inner):
  """Recursive function to draw concentric circles"""
  if radius < gap:
   return 0
  else:
   draw_disc(radius, color_out)
   lt(90); fd(gap); rt(90)
   num = concentric_circles(radius-gap, gap, color_in, color_out)
   return 1 + num
```



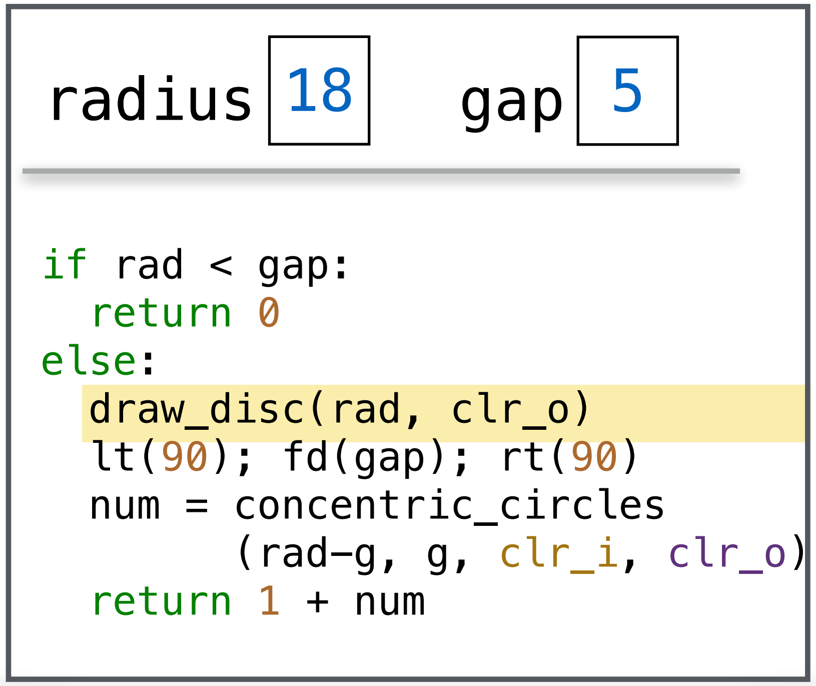
```python
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```

```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

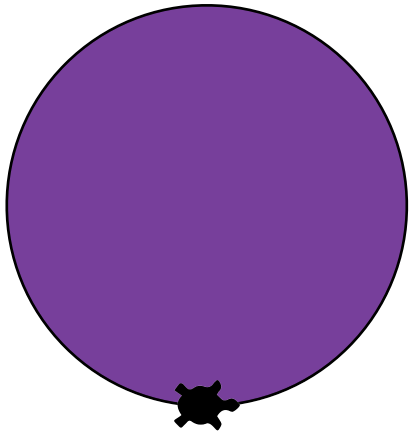
```
>>> concentric_circles(18, 5, "purple", "gold")
```

contrc_circles(18,5,'p','g')



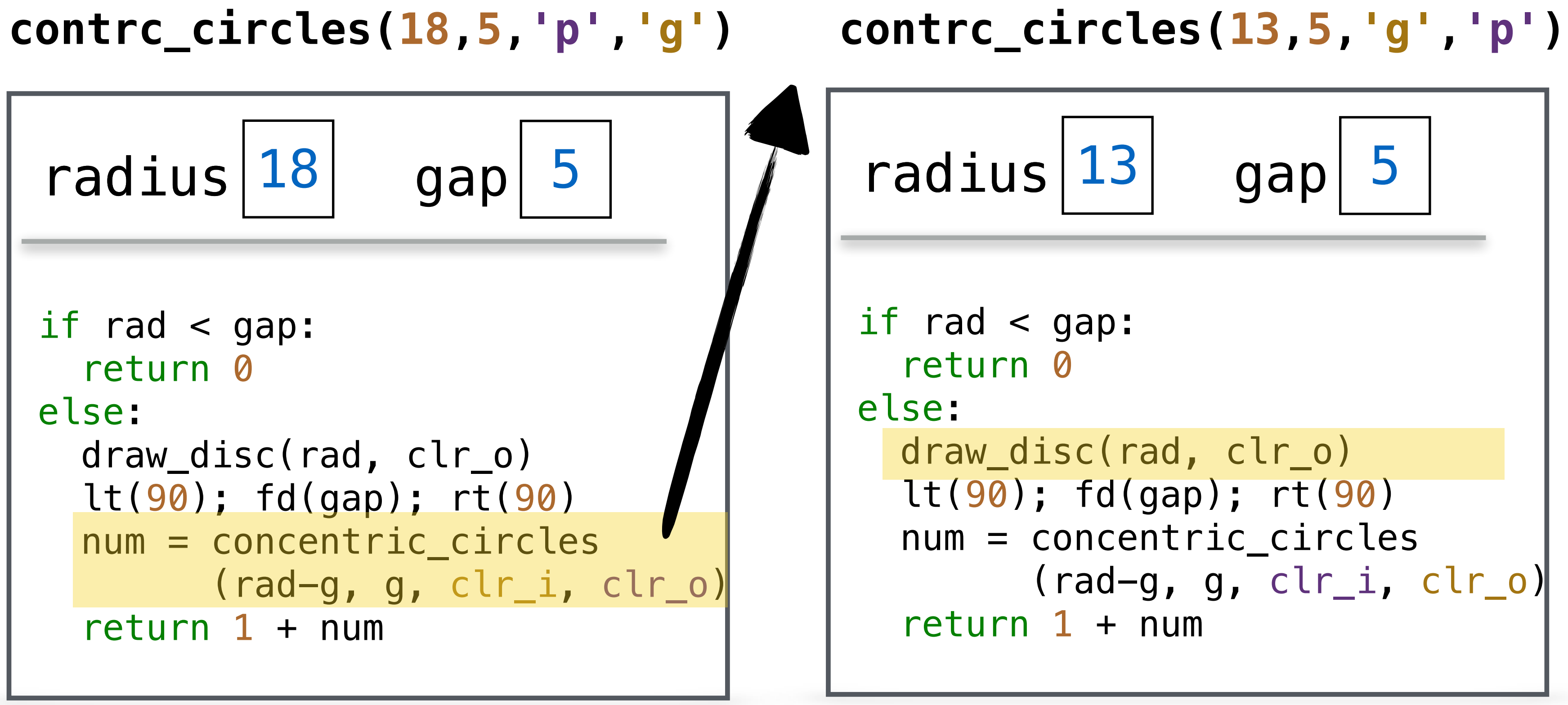
radius 18 gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```



```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



contrc_circles(18,5,'p','g')

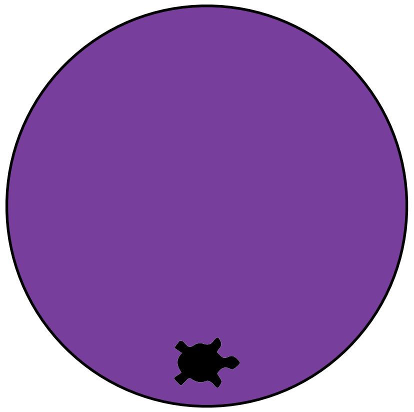
radius 18 gap 5

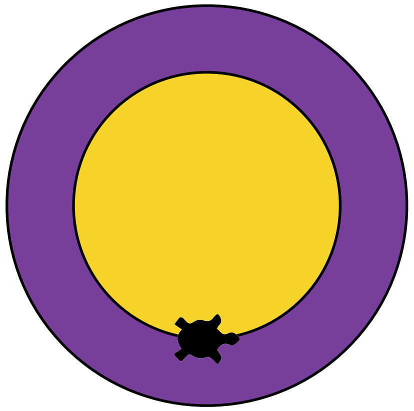
```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

radius 13 gap 5

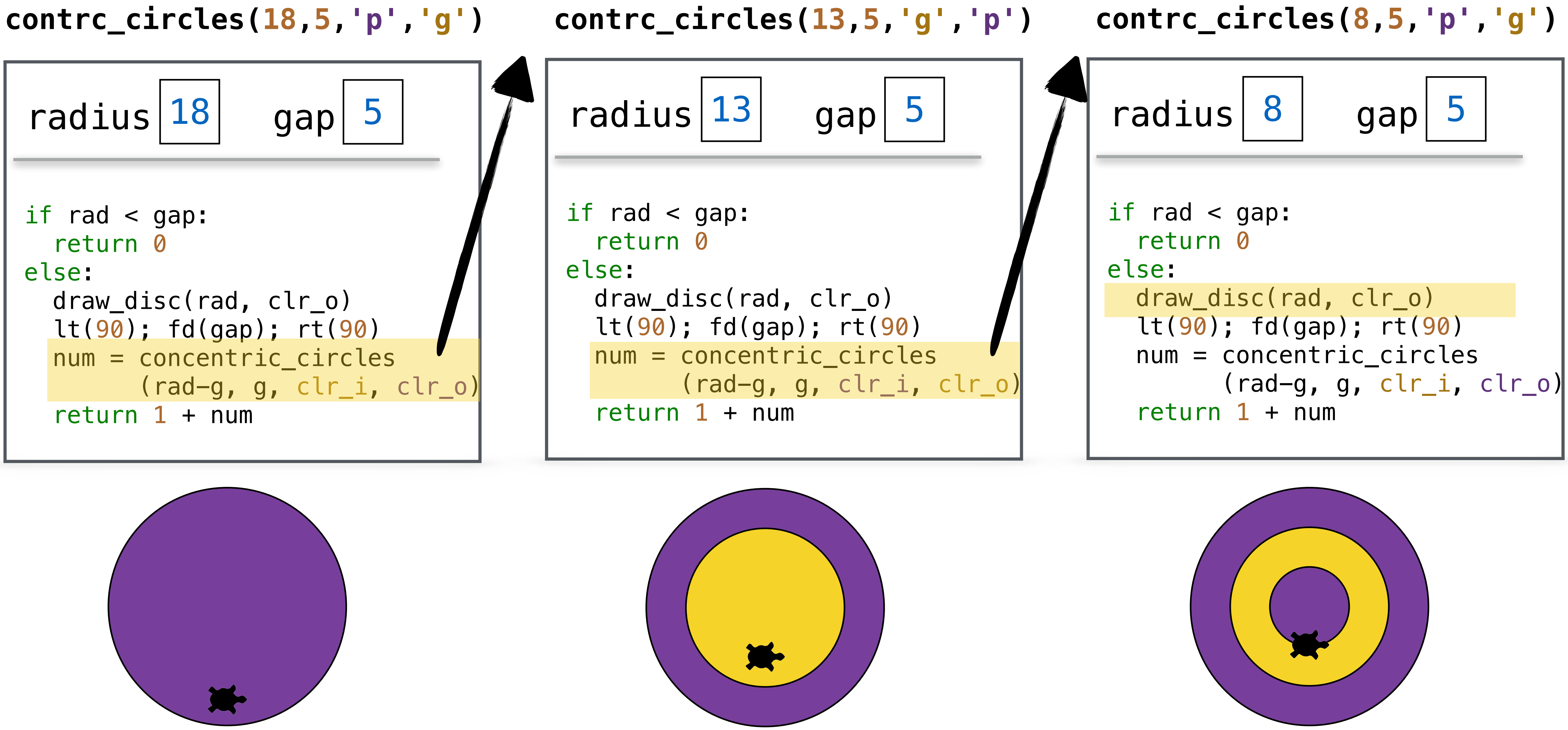
```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```





```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



```
contrc_circles(18,5,'p','g')

radius 18    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(13,5,'g','p')

radius 13    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(8,5,'p','g')

radius 8    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```python
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```

contrc_circles(3,5,'g','p')

```
radius 3     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(18,5,'p','g')

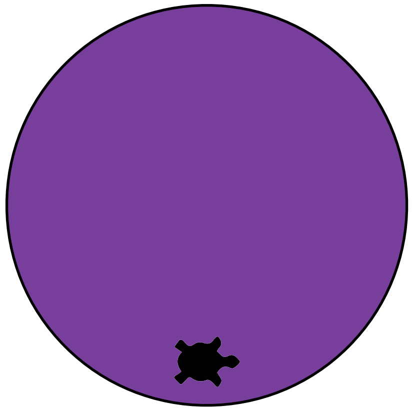
```
radius 18     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

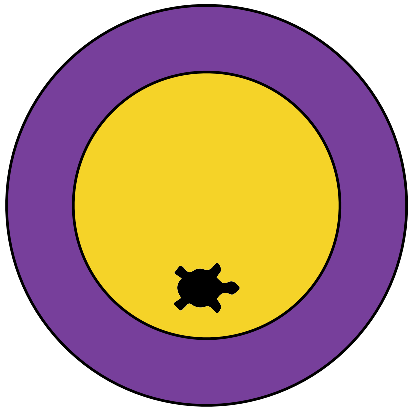
contrc_circles(13,5,'g','p')

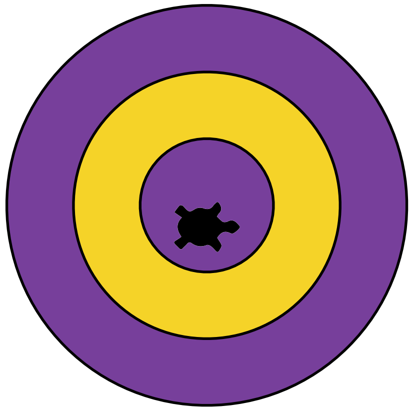
```
radius 13     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(8,5,'p','g')

```
radius 8     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```







```python
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```

contrc_circles(3,5,'g','p')

radius 3   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(18,5,'p','g')

radius 18   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

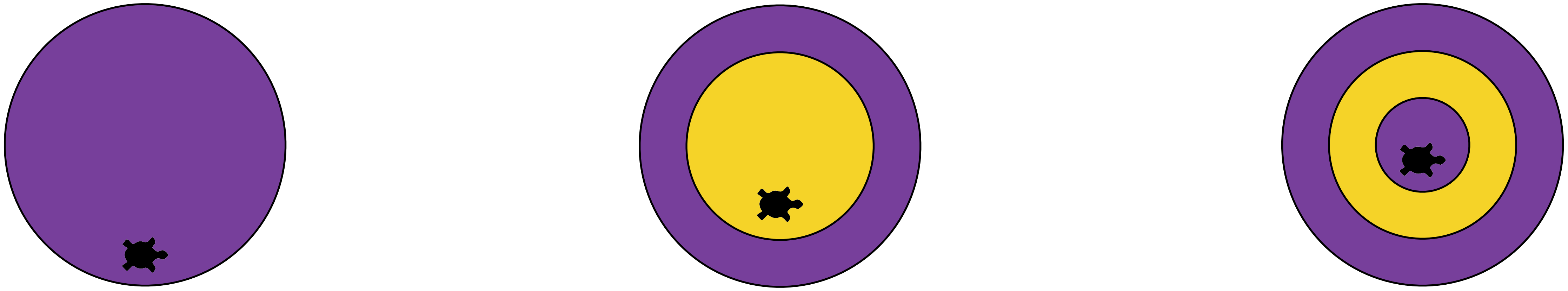
radius 13   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(8,5,'p','g')

radius 8   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = c 0 tric_circles
        (  , g, clr_i, clr_o)
  return 1 + num
```



```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```

contrc_circles(3,5,'g','p')

```
radius 3     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(18,5,'p','g')

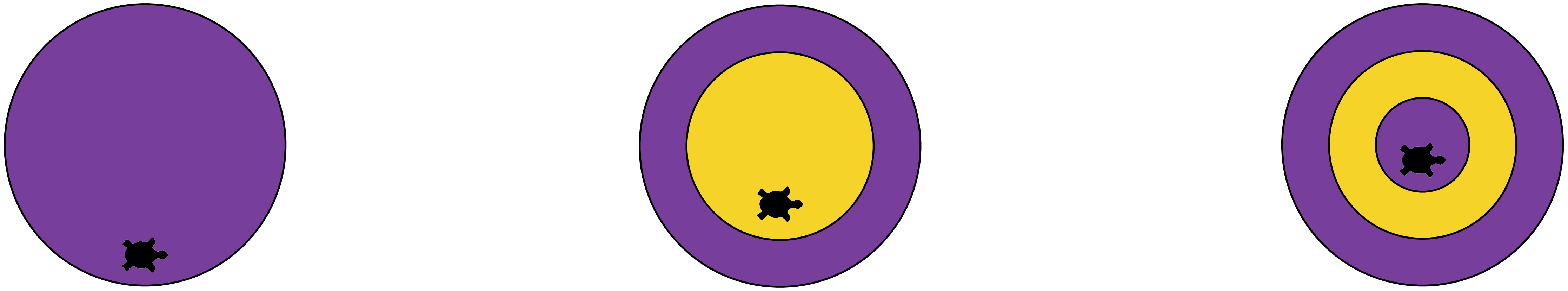
```
radius 18     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

```
radius 13     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

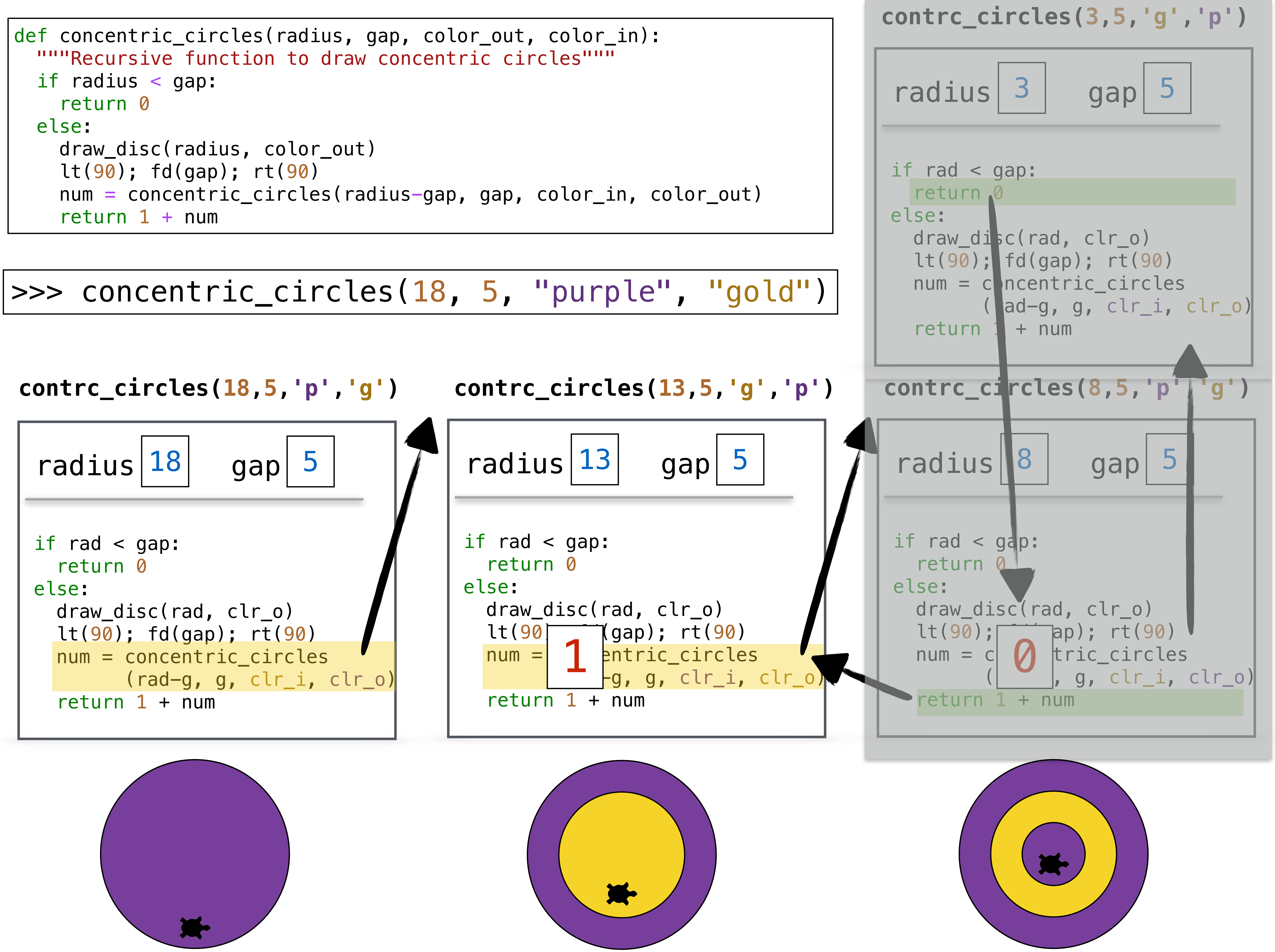
contrc_circles(8,5,'p','g')

```
radius 8     gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90);     ap); rt(90)
  num = c  0  tric_circles
        (   , g, clr_i, clr_o)
  return 1 + num
```



```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



contrc_circles(18,5,'p','g')

radius 18   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

radius 13   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90      (gap); rt(90)
  num = 1   entric_circles
            -g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(8,5,'p','g')

radius 8   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90);     ap); rt(90)
  num = c  0  tric_circles
        (    , g, clr_i, clr_o)
  return 1 + num
```

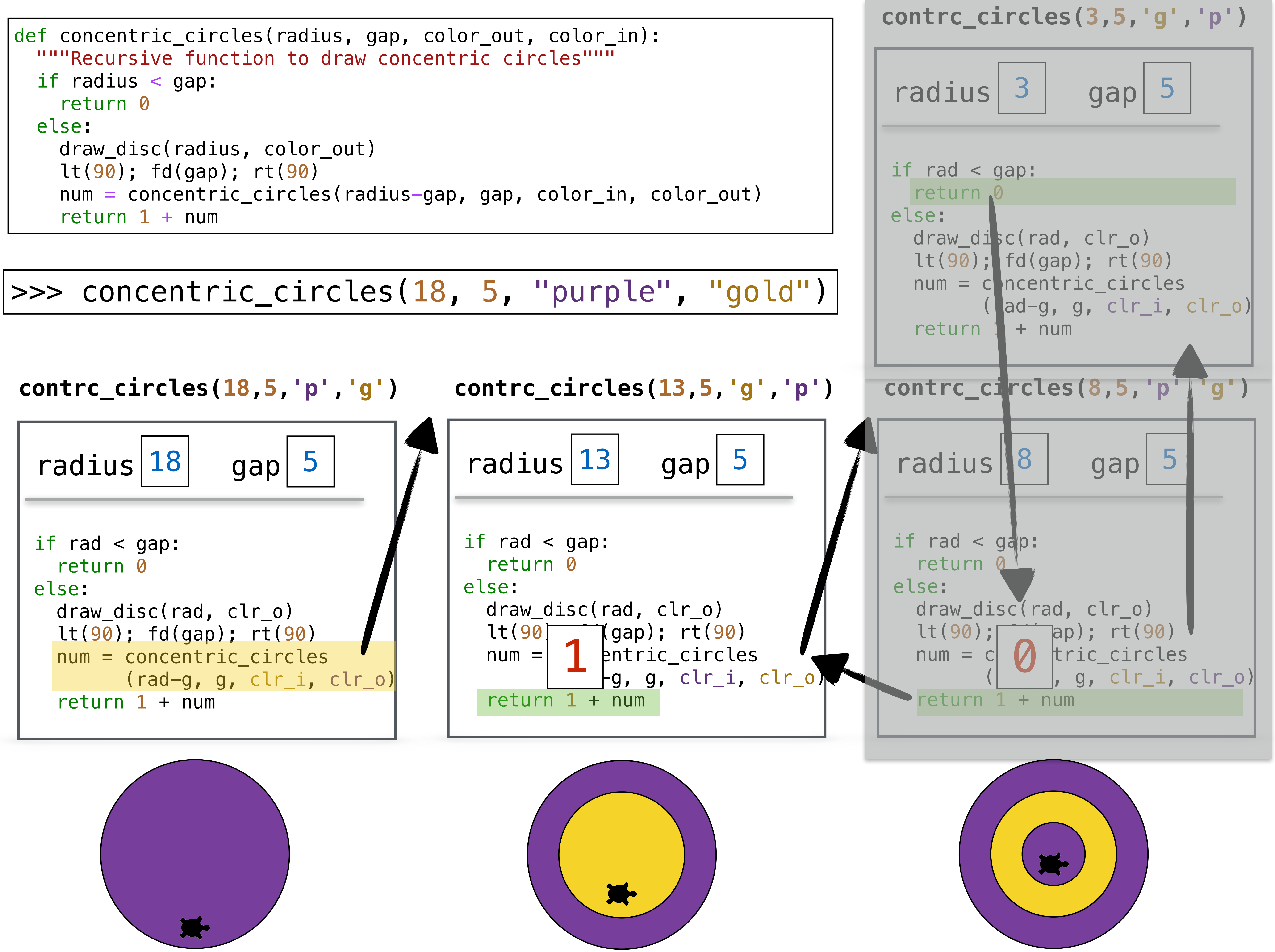
contrc_circles(3,5,'g','p')

radius 3   gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```python
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



contrc_circles(18,5,'p','g')

radius 18 gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

radius 13 gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

1

contrc_circles(8,5,'p','g')

radius 8 gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

0

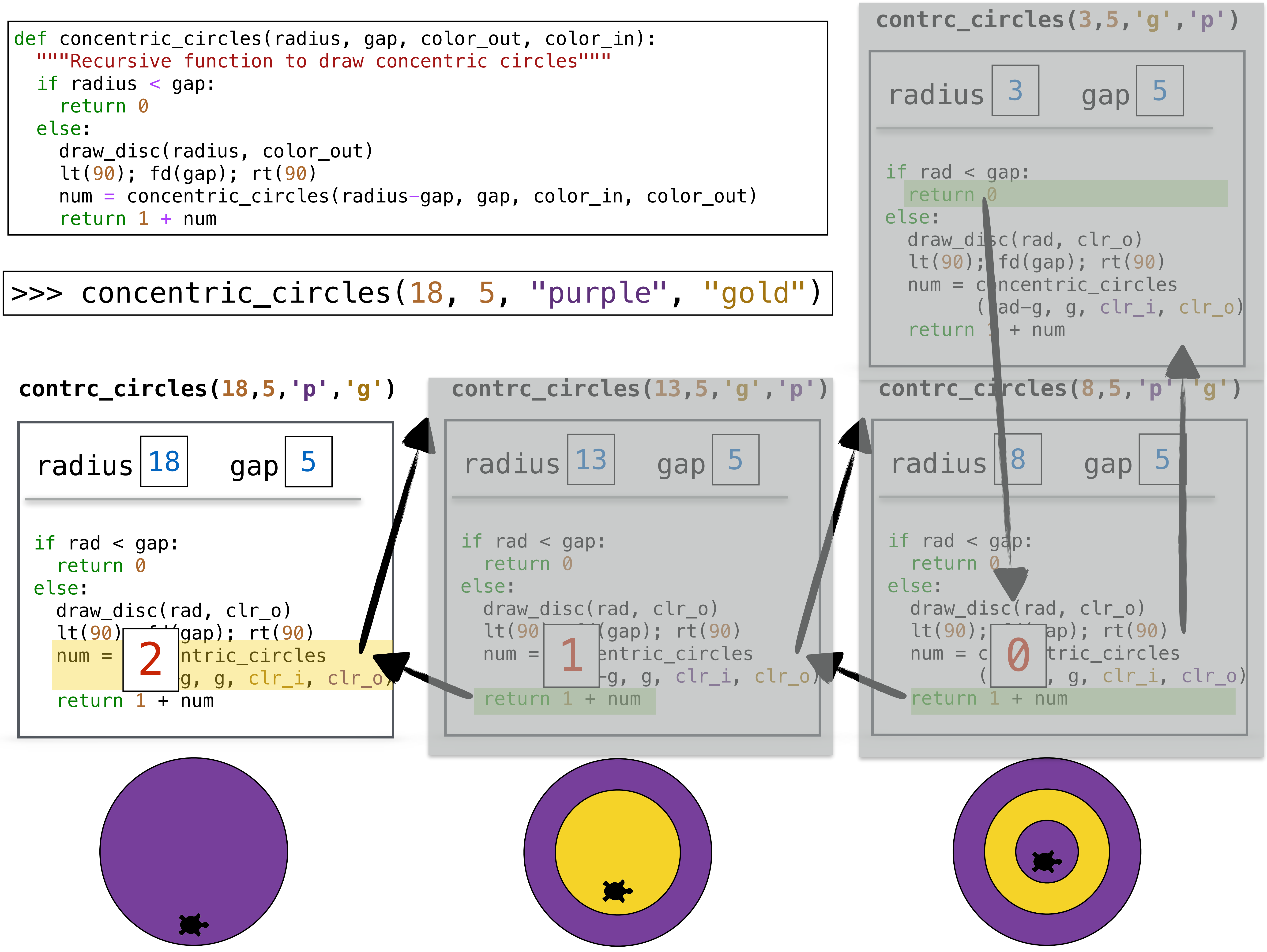
contrc_circles(3,5,'g','p')

radius 3 gap 5

```
if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



contrc_circles(18,5,'p','g')

```
radius 18    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90)    gap); rt(90)
  num = 2 ntric_circles
          g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(13,5,'g','p')

```
radius 13    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90 1 (gap); rt(90)
  num =   entric_circles
          -g, g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(8,5,'p','g')

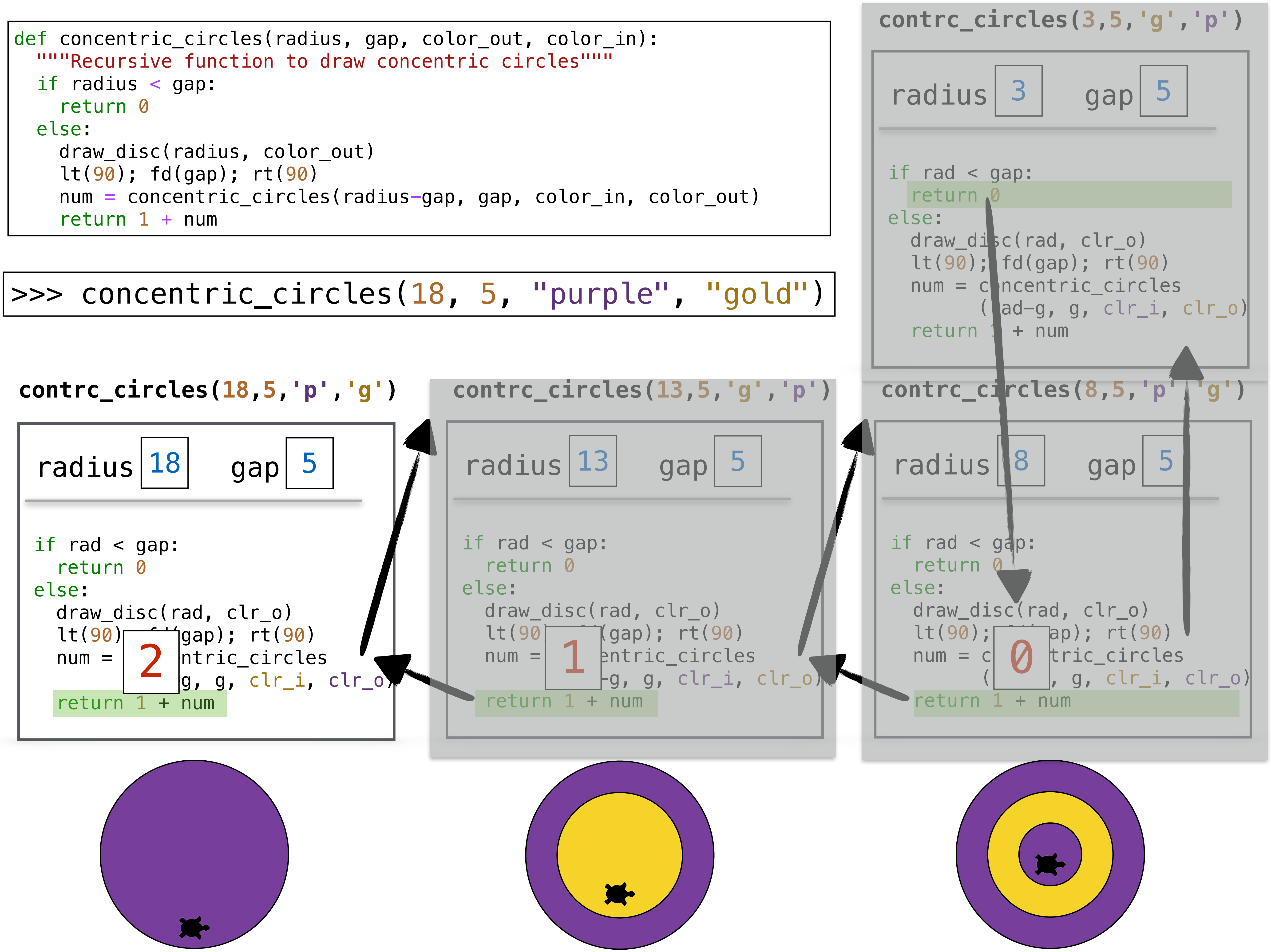
```
radius 8    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); 0 ap); rt(90)
  num = c   tric_circles
        (   , g, clr_i, clr_o)
  return 1 + num
```

contrc_circles(3,5,'g','p')

```
radius 3    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> concentric_circles(18, 5, "purple", "gold")
```



```
contrc_circles(18,5,'p','g')
radius 18   gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90)      gap); rt(90)
  num =  2   ntric_circles
             g, g, clr_i, clr_o)
  return 1 + num
```

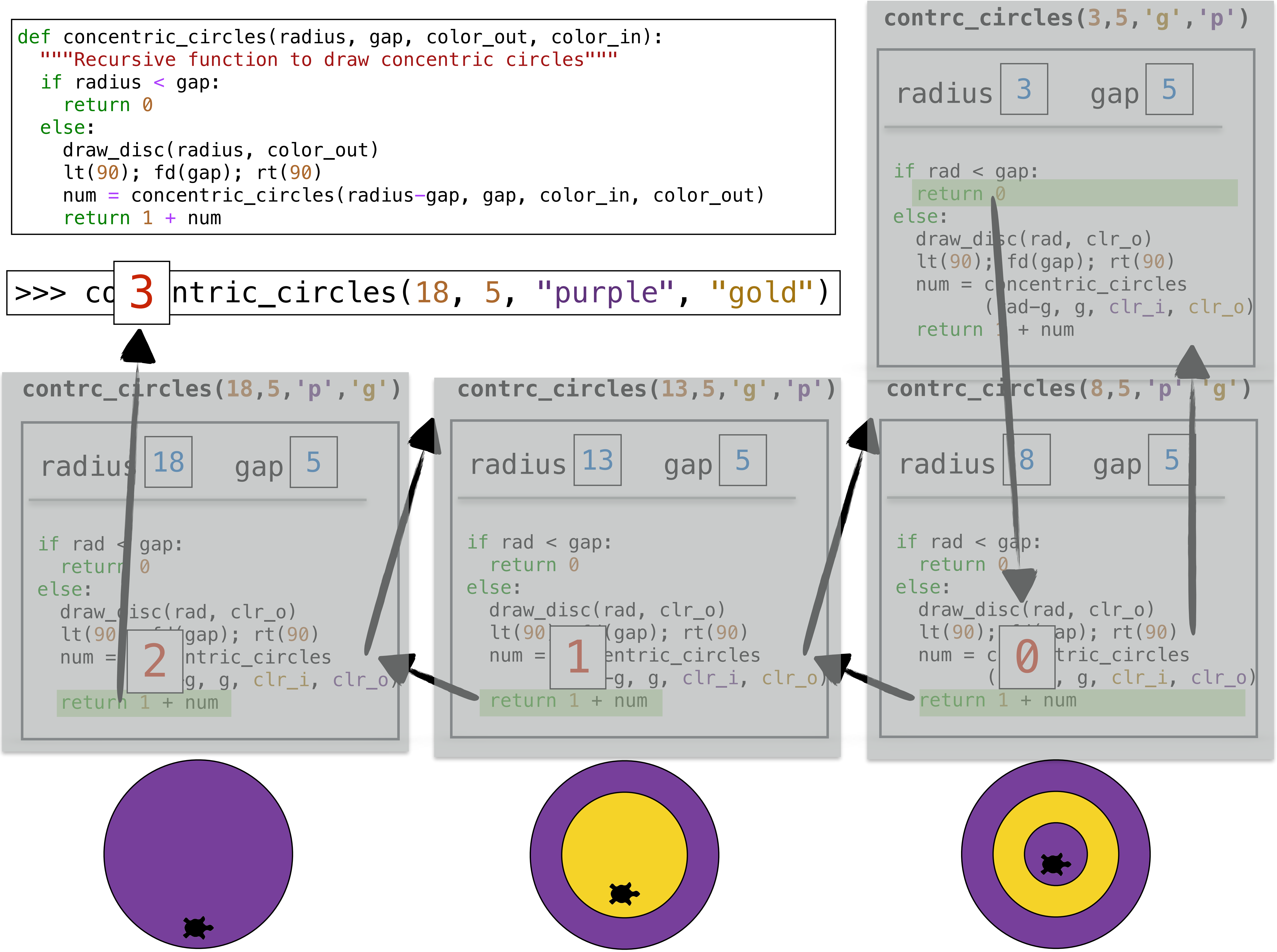
```
contrc_circles(13,5,'g','p')
radius 13   gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90      (gap); rt(90)
  num =  1   entric_circles
             -g, g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(8,5,'p','g')
radius 8   gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90);      ap); rt(90)
  num = c  0  tric_circles
         (    , g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(3,5,'g','p')
radius 3   gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
          (rad-g, g, clr_i, clr_o)
  return 1 + num
```

```
def concentric_circles(radius, gap, color_out, color_in):
  """Recursive function to draw concentric circles"""
  if radius < gap:
    return 0
  else:
    draw_disc(radius, color_out)
    lt(90); fd(gap); rt(90)
    num = concentric_circles(radius-gap, gap, color_in, color_out)
    return 1 + num
```

```
>>> co 3 ntric_circles(18, 5, "purple", "gold")
```



```
contrc_circles(18,5,'p','g')

radius 18    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90  2  gap); rt(90)
  num =  2  ntric_circles
            g, g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(13,5,'g','p')

radius 13    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90  1  (gap); rt(90)
  num =  1  entric_circles
            -g, g, clr_i, clr_o)
  return 1 + num
```

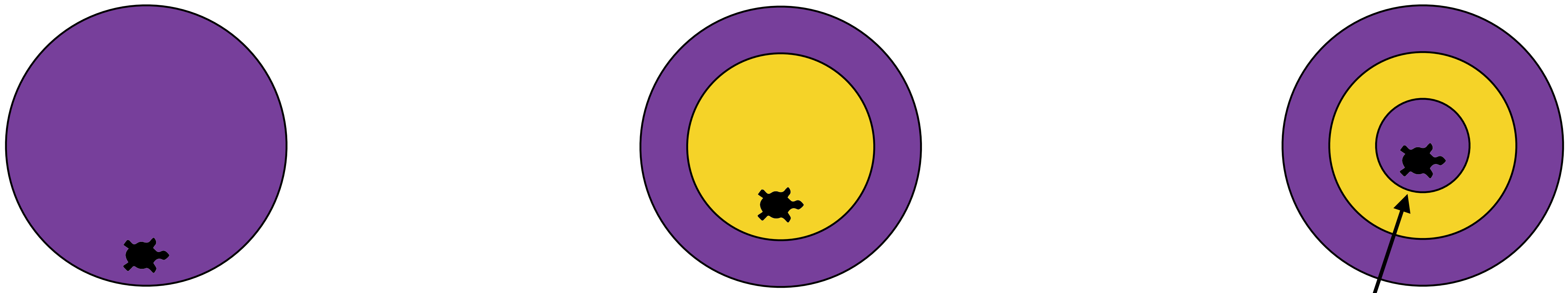
```
contrc_circles(8,5,'p','g')

radius 8    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90);  0  ap); rt(90)
  num = c  0  tric_circles
         (   , g, clr_i, clr_o)
  return 1 + num
```

```
contrc_circles(3,5,'g','p')

radius 3    gap 5

if rad < gap:
  return 0
else:
  draw_disc(rad, clr_o)
  lt(90); fd(gap); rt(90)
  num = concentric_circles
        (rad-g, g, clr_i, clr_o)
  return 1 + num
```

Invariance of Functions

- Where does the turtle end up in this example with `concentric_circles_color`?

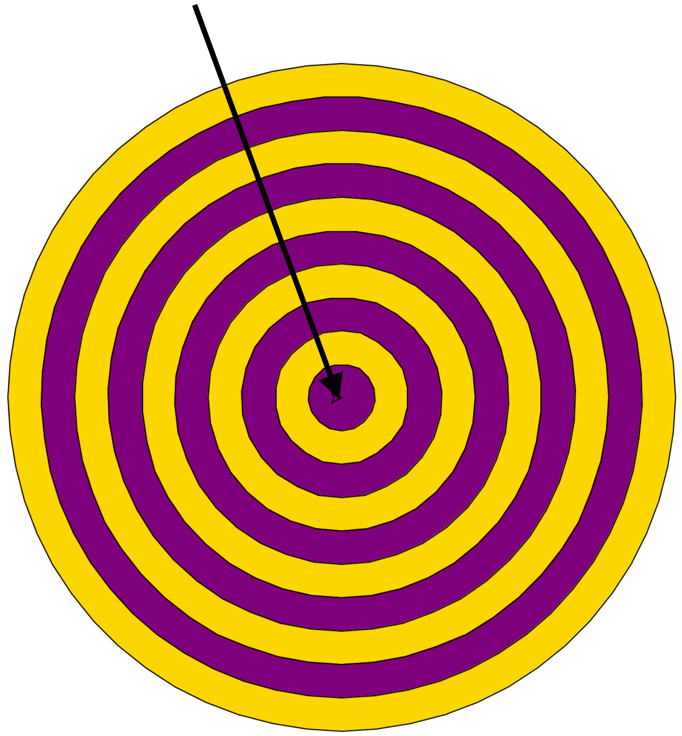
```
concentric_circles(18, 5, 'purple', 'gold')
```



- The turtle does not end where it starts

turtle ends near center

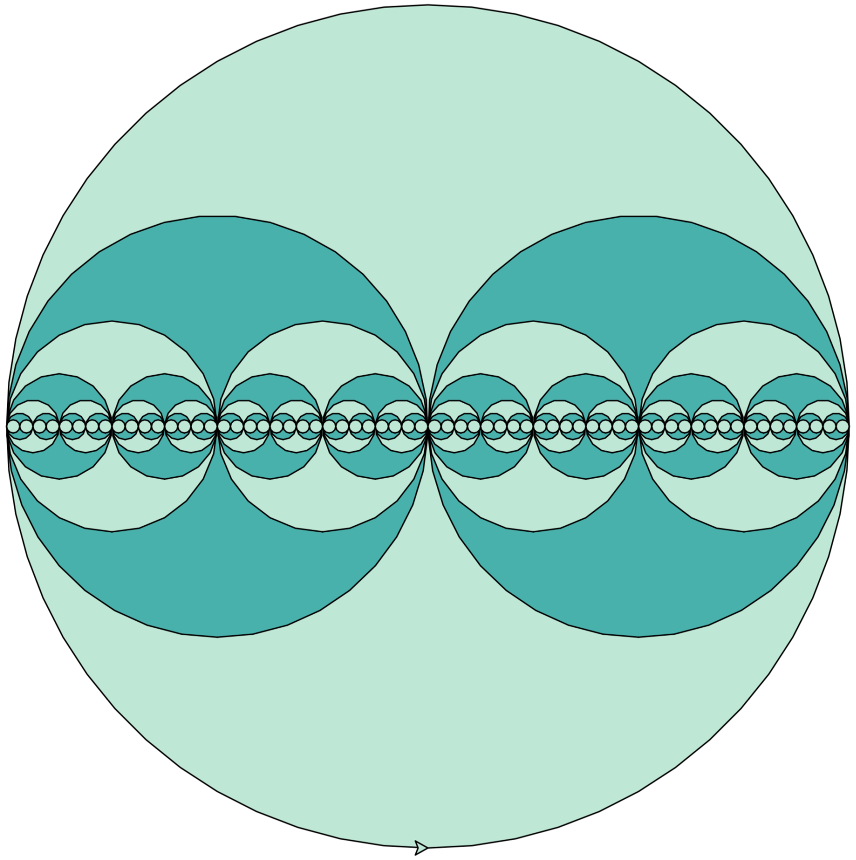
```
def concentric_circles_color(radius, gap, color_outer, color_inner):
    """
    Recursive function to draw concentric circles with
    alternating colors
    """
    # base case, don't draw anything, return 0
    if radius < gap:
        return 0
    else:
        draw_disc(radius, color_outer)
        lt(90)
        fd(gap)
        rt(90)
        num = concentric_circles_color(radius-gap, gap, color_inner, color_outer)
        return 1 + num
```



Example: Nested Circles

Invariance of Recursive Functions

- Let's do an example with multiple recursive calls: nested circles

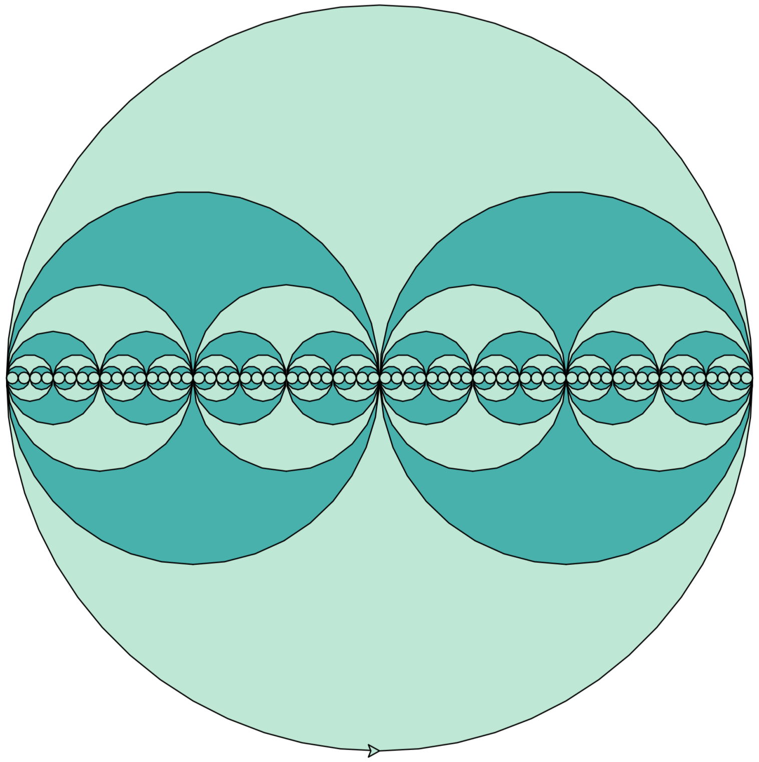


Multiple Recursive Calls

- **Example:** Nested circles function definition

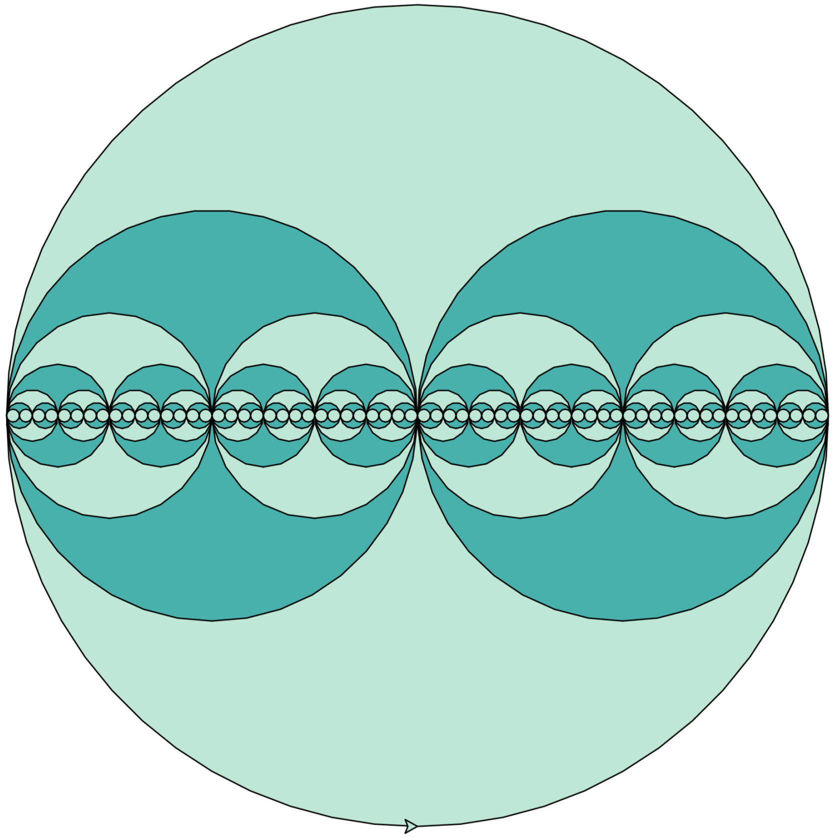
`nested_circles(radius, min_radius, color_out, color_alt)`

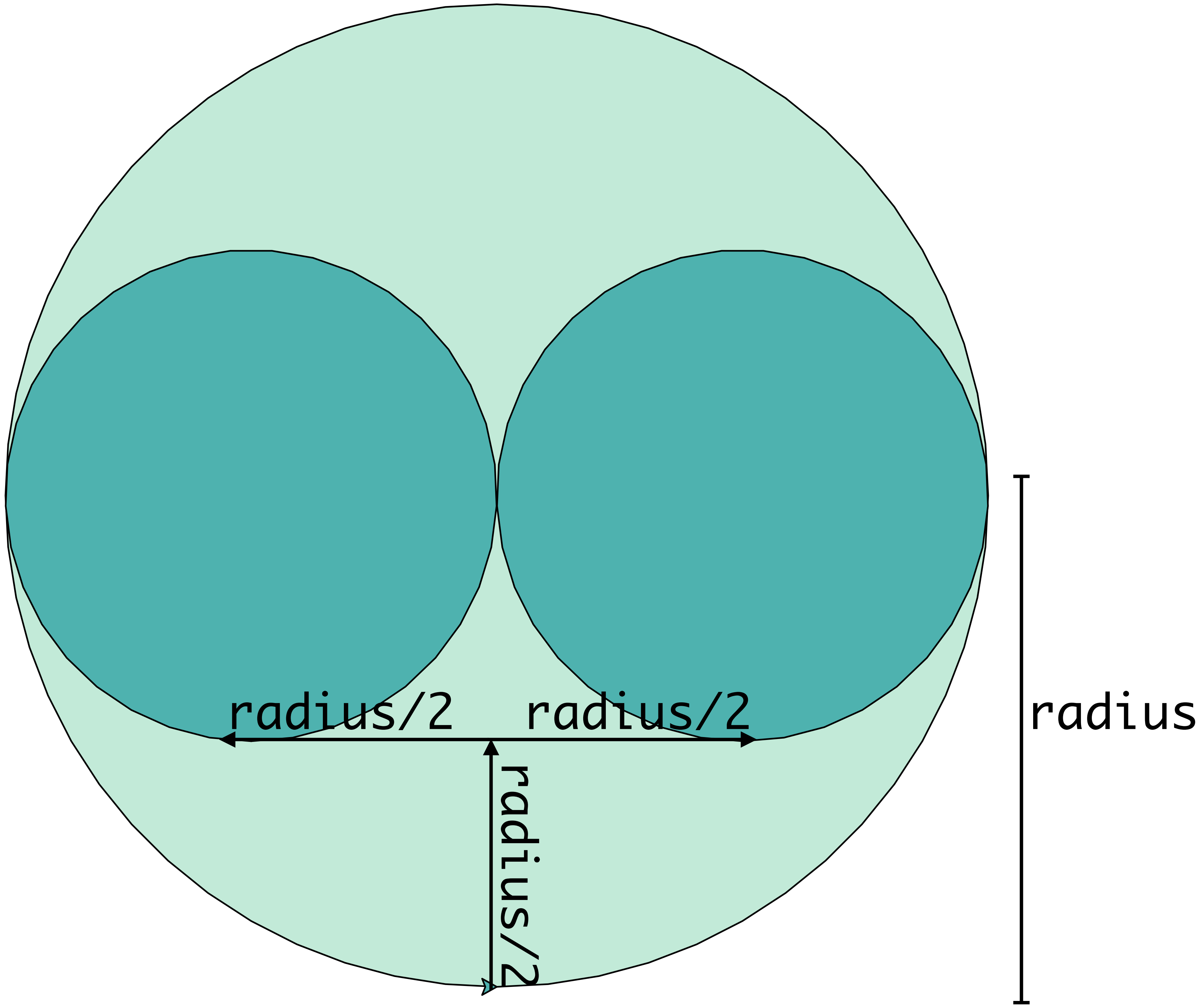
  - `radius`: radius of the outermost circle
  - `min_radius`: minimum radius of any circle
  - `color_out`: color of the outermost circle
  - `color_alt`: color that alternates with colorOut



Nested Circles

- **Base case?**
  - When radius becomes less than min_radius
  - Don't draw anything return 0
- **Recursive case**
  - Draw the outer circle, add one to total
  - Position turtle for recursive calls



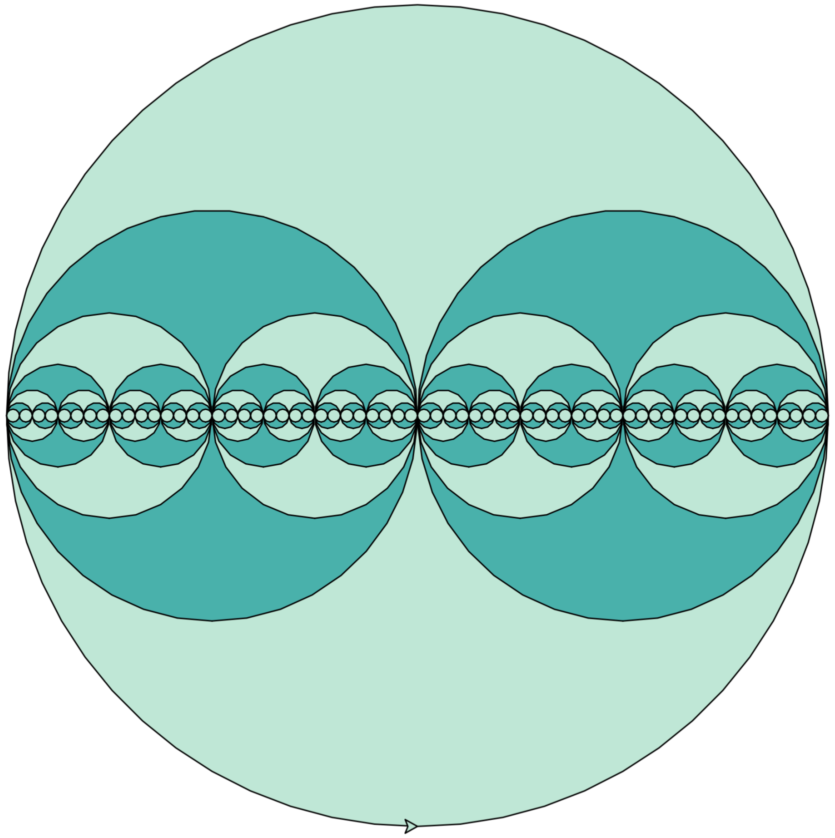


Starting position of turtle

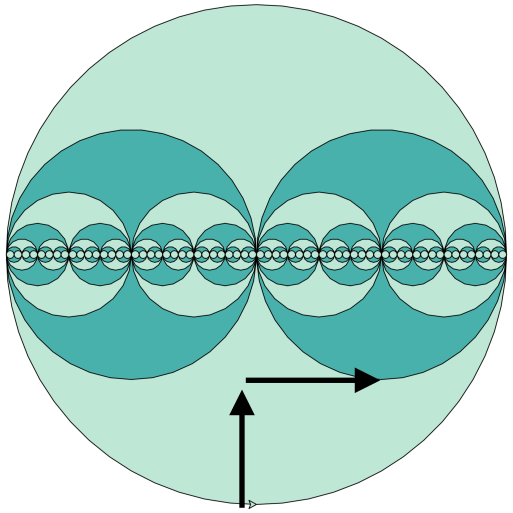
`nested_circles(300, 150)`

Nested Circles

- **Base case?**
  - When radius becomes less than minRadius
  - Don't draw anything return 0
- **Recursive case**
  - Draw the outer circle, add one to total
  - Position turtle for recursive calls
  - How many recursive calls do we need?
    - Two! Right subcircle and left subcircle



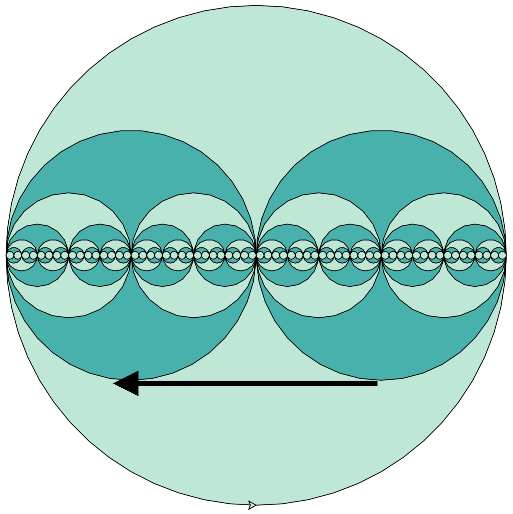
Nested Circles



- **Recursive case**
  - Draw the outer circle, add one to total
  - Position turtle for right recursive subcircle

```python
def nested_circles(radius, min_radius, color_out, color_alt):
    if radius < min_radius:
        return 0
    else:
        # contribute to the solution
        draw_disc(radius, color_out)

        # save half of radius
        half_radius = radius/2

        # position the turtle to draw right subcircle
        lt(90); fd(half_radius); rt(90); fd(half_radius)

        # draw right subcircle recursively
        right = nested_circles(half_radius, min_radius, color_alt, color_out)
```

Nested Circles



- **Recursive case**
    - Move the turtle to draw left subcircle recursively
    - (continued from previous slide)

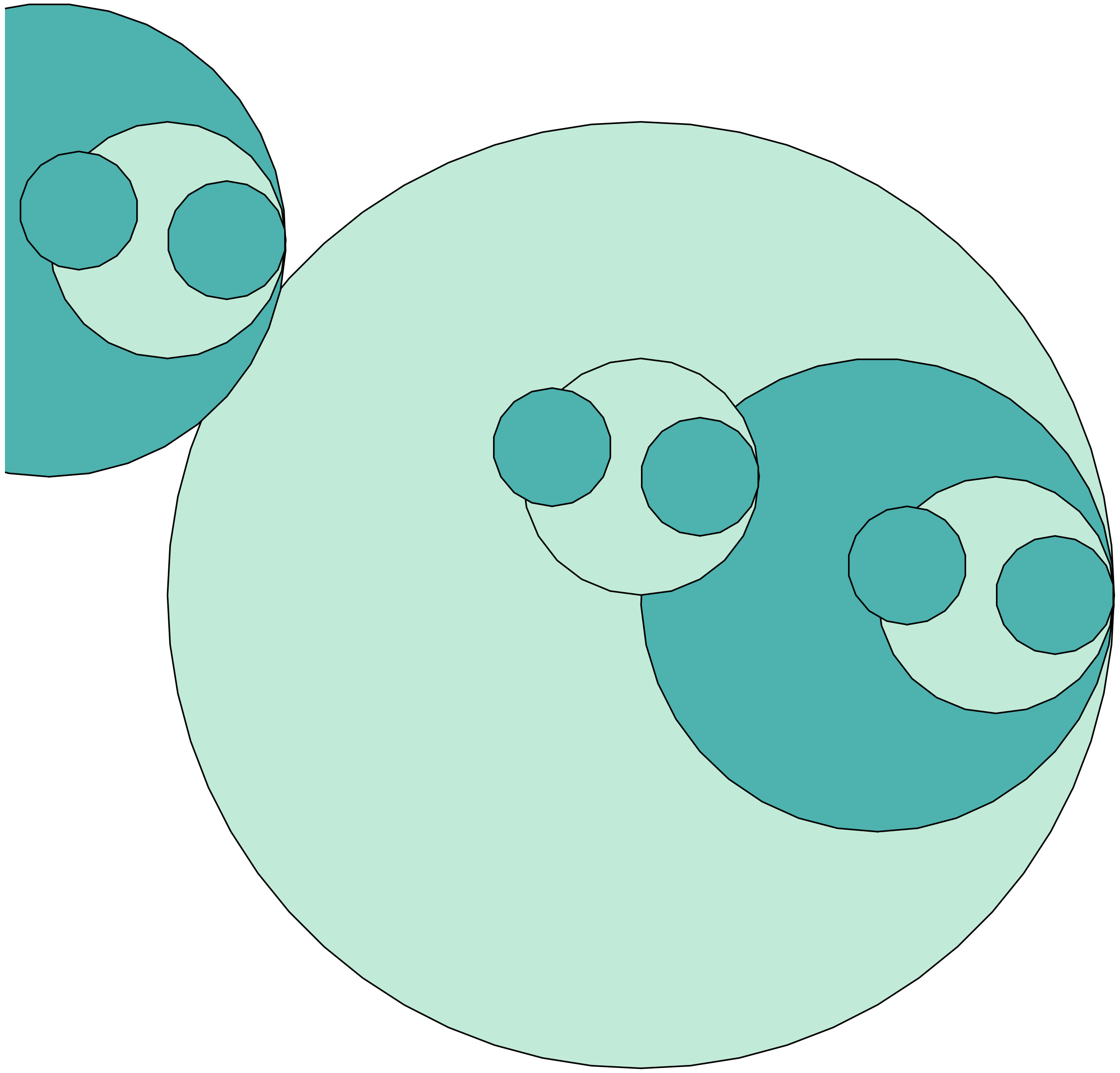
```
    # draw right subcircle recursively
    right = nested_circles(half_radius, min_radius, color_alt, color_out)

    # position turtle for left subcircle
    bk(radius)

    # draw left subcircle recursively
    left = nested_circles(half_radius, min_radius, color_alt, color_out)

    # add one to our count of subcircles
    return 1 + num
```

Nested Circles

- **Recursive case**
  - Are we done? Let’s try it!

Nested Circles

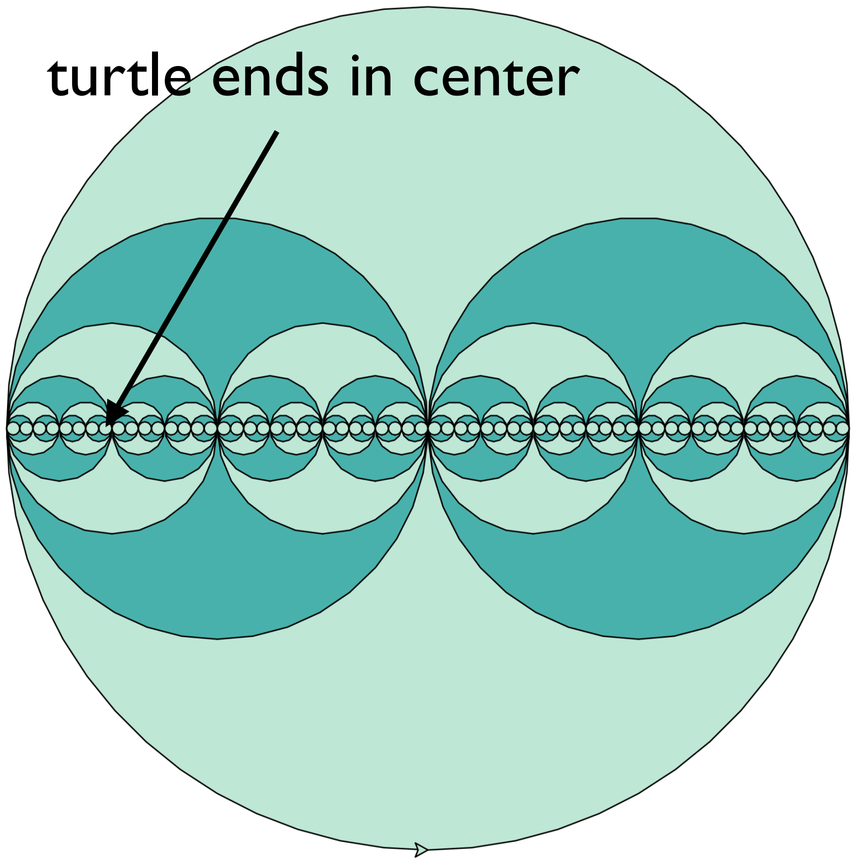
- **Recursive case**
  - What happened?!
  - We made assumptions about where the turtle started, that wasn't true!
  - Need turtle to *end* where it *started*
  - This is called ***position invariance***



Invariance of Functions

- A function is **invariant** if the state of the object is the same *before* and *after* the function is invoked
- Right now our `nested_circles` function is not invariant with respect to the position of the turtle
  - That is, the turtle does not end were it starts
- How can we make it invariant by returning the turtle to starting position?

```
def nested_circles(radius, min_radius, color_out, color_alt):
    if radius < min_radius:
        return 0
    else:
        draw_disc(radius, color_out)
        h_r = radius/2

        lt(90); fd(h_r); rt(90); fd(h_r)

        right = nested_circles(h_r, min_radius, color_alt, color_out)

        bk(radius)

        left = nested_circles(h_r, min_radius, color_alt, color_out)

        fd(h_r);  lt(90);  bk(h_r);  rt(90)
        return 1 + right + left
```

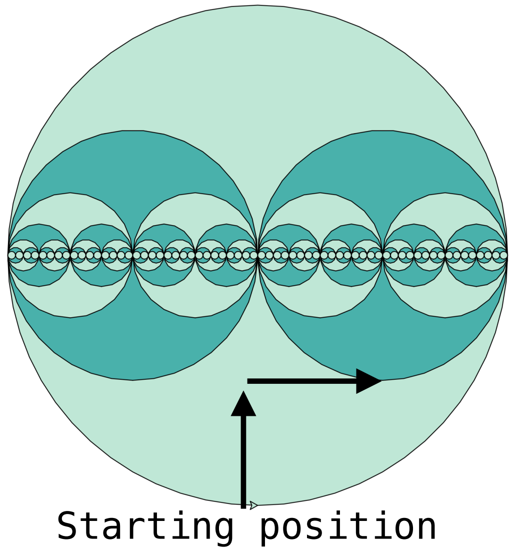


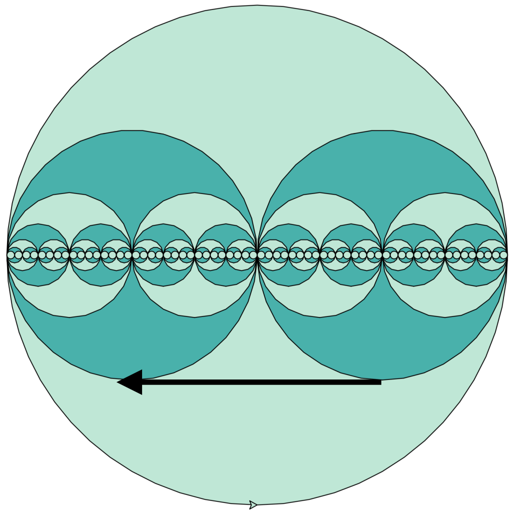
turtle ends in center

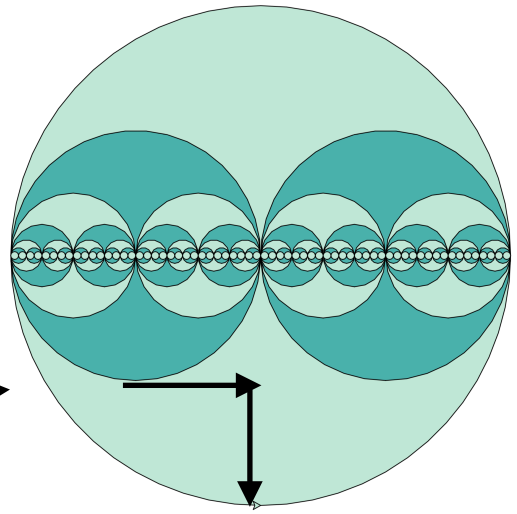
Maintaining Invariance

- Any turtle movements that happen before the recursive call should be "undone" after the recursive call to maintain proper invariance
- **Rule of thumb:** always return turtle to its starting position

```
def nested_circles(radius, min_radius, color_out, color_alt):
    if radius < min_radius:
        return 0
    else:
        # contribute to the solution
        draw_disc(radius, color_out)

        # save half of radius
        half_radius = radius/2

        # position the turtle to draw right subcircle
        lt(90); fd(half_radius); rt(90); fd(half_radius)

        # draw right subcircle recursively
        right = nested_circles(half_radius, min_radius, color_alt, color_out)

        # position turtle for left subcircle
        bk(radius)

        # draw left subcircle recursively
        left = nested_circles(half_radius, min_radius, color_alt, color_out)

        # bring turtle back to start position
        fd(half_radius);  lt(90);  bk(half_radius);  rt(90)

        # return total number of circles drawn
        return 1 + right + left
```

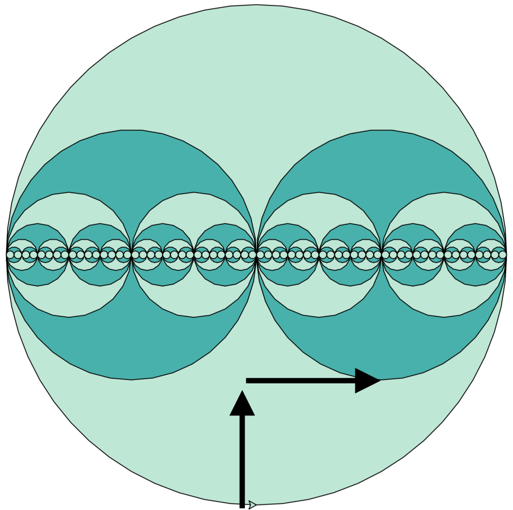




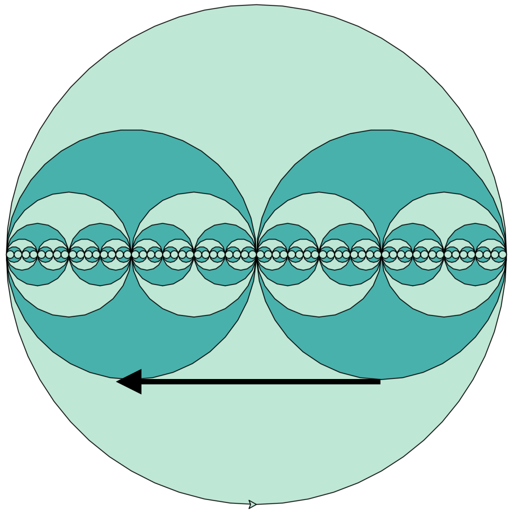


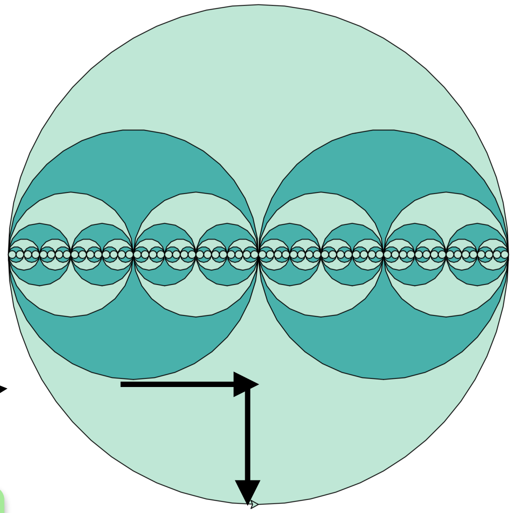
Maintaining Invariance

- Move turtle back to starting position to maintain **invariance**

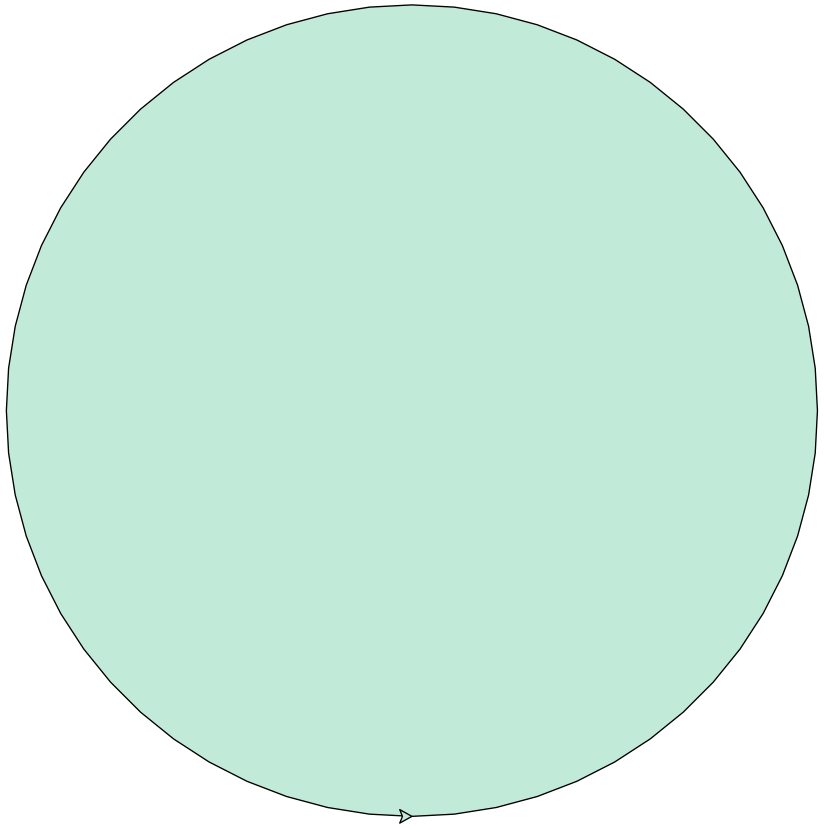


```
def nested_circles(radius, min_radius, color_out, color_alt):
    if radius < min_radius:
        return 0
    else:
        # contribute to the solution
        draw_disc(radius, color_out)

        # save half of radius
        half_radius = radius/2

        # position the turtle to draw right subcircle
        lt(90); fd(half_radius); rt(90); fd(half_radius)

        # draw right subcircle recursively
        right = nested_circles(half_radius, min_radius, color_alt, color_out)

        # position turtle for left subcircle
        bk(radius)

        # draw left subcircle recursively
        left = nested_circles(half_radius, min_radius, color_alt, color_out)

        # bring turtle back to start position
        fd(half_radius);  lt(90);  bk(half_radius);  rt(90)

        # return total number of circles drawn
        return 1 + right + left
```

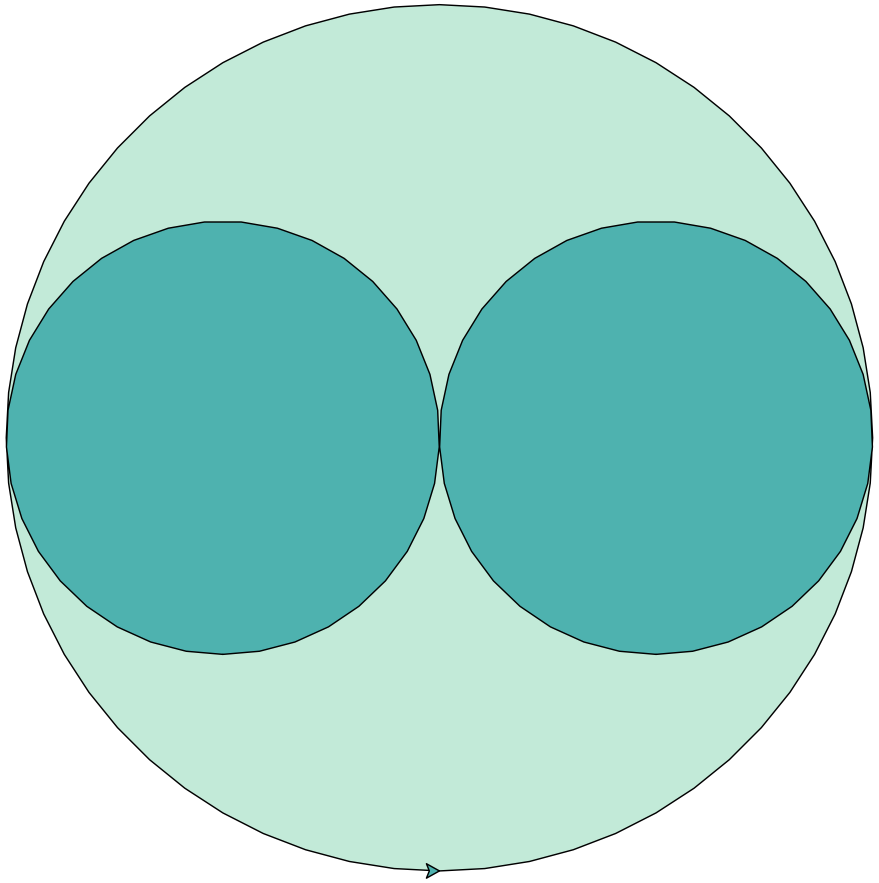




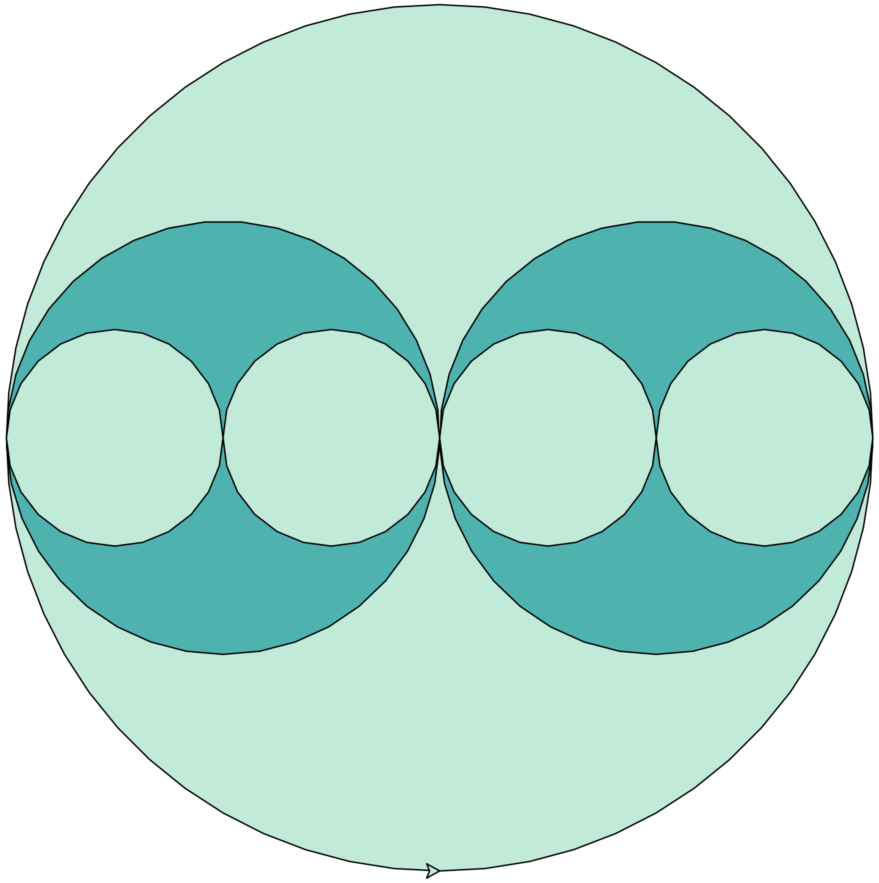
Maintain invariance



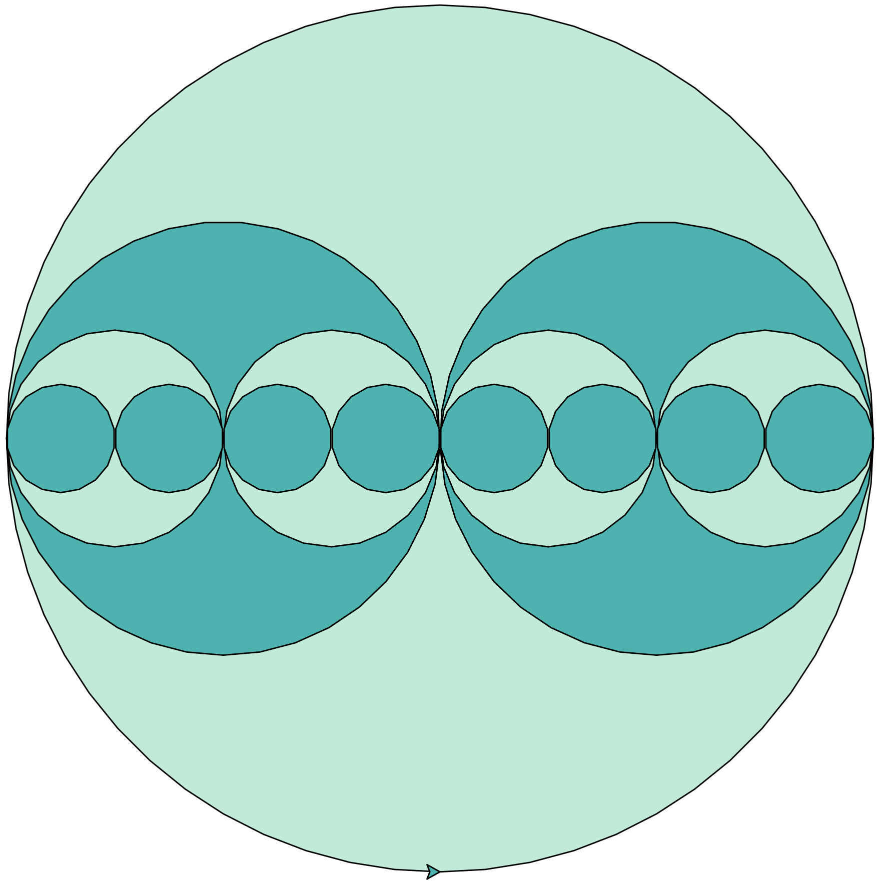
nestedCircles(300, 300)



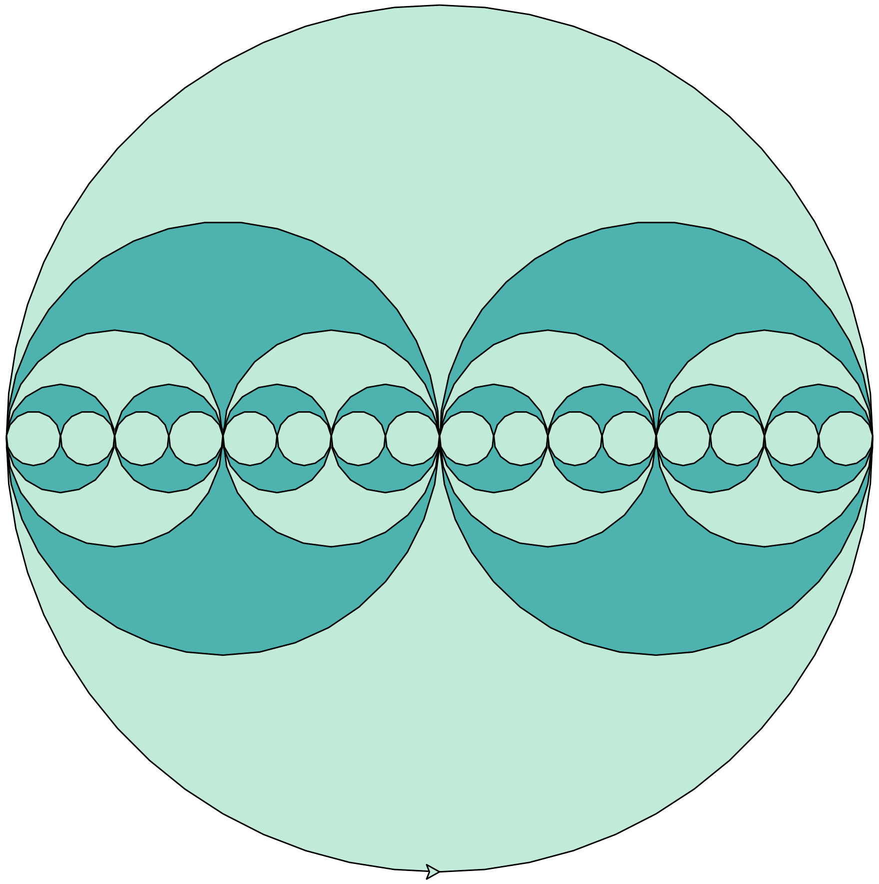
nestedCircles(300, 150)



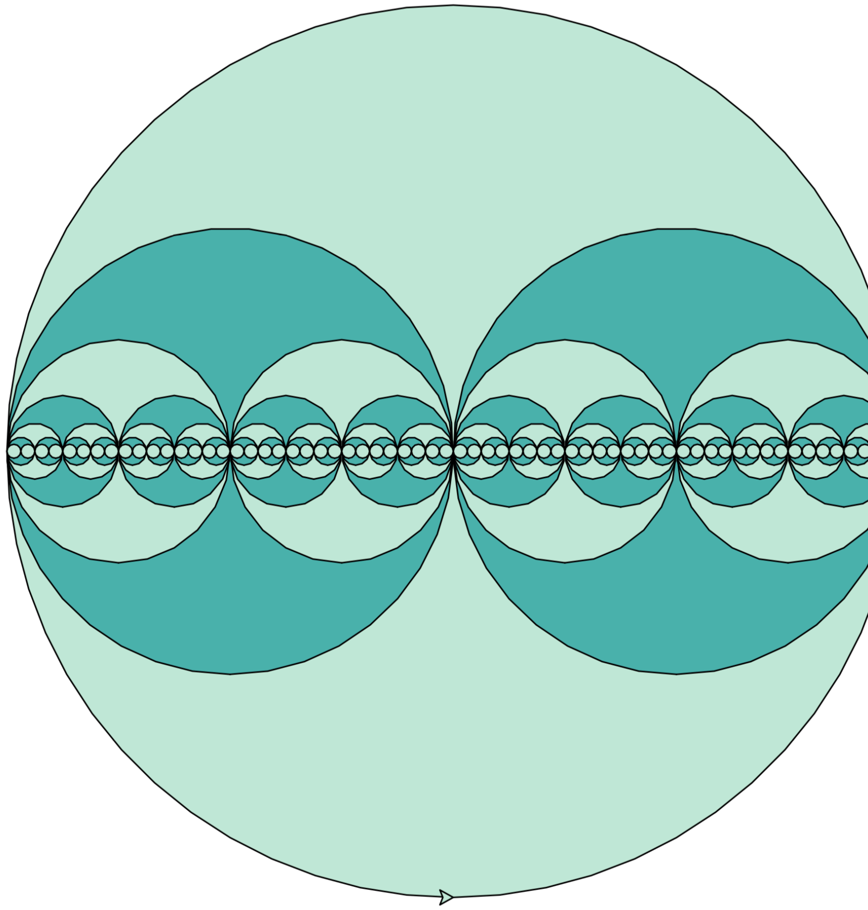
nestedCircles(300, 75)



nestedCircles(300, 37)



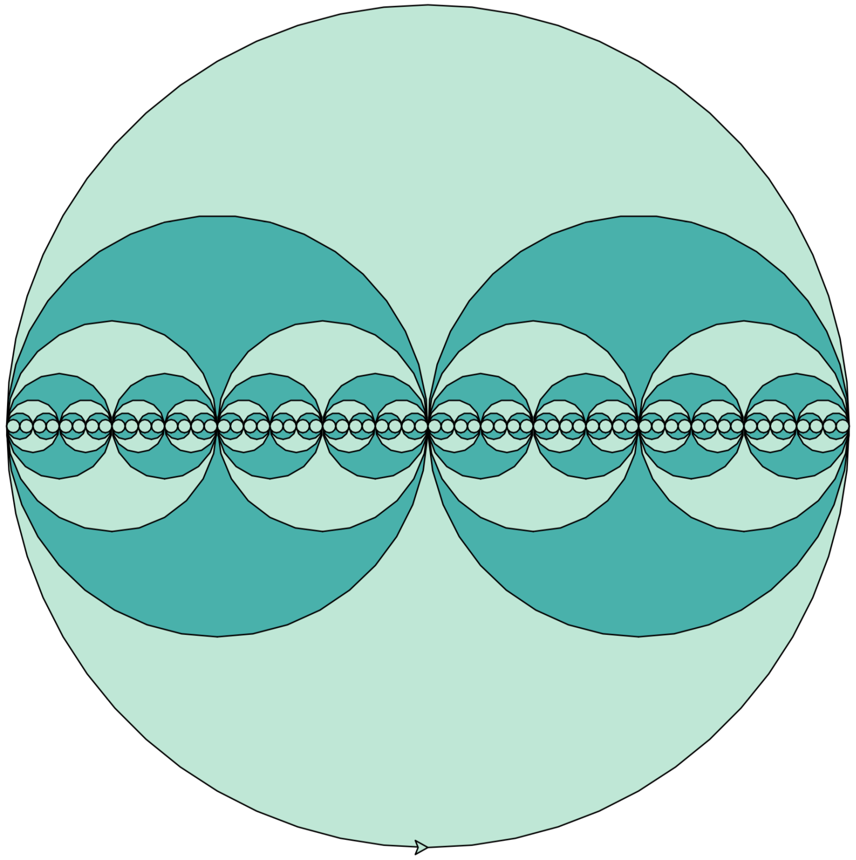
nestedCircles(300, 9)



nestedCircles(300, 2)

Invariance of Recursive Functions

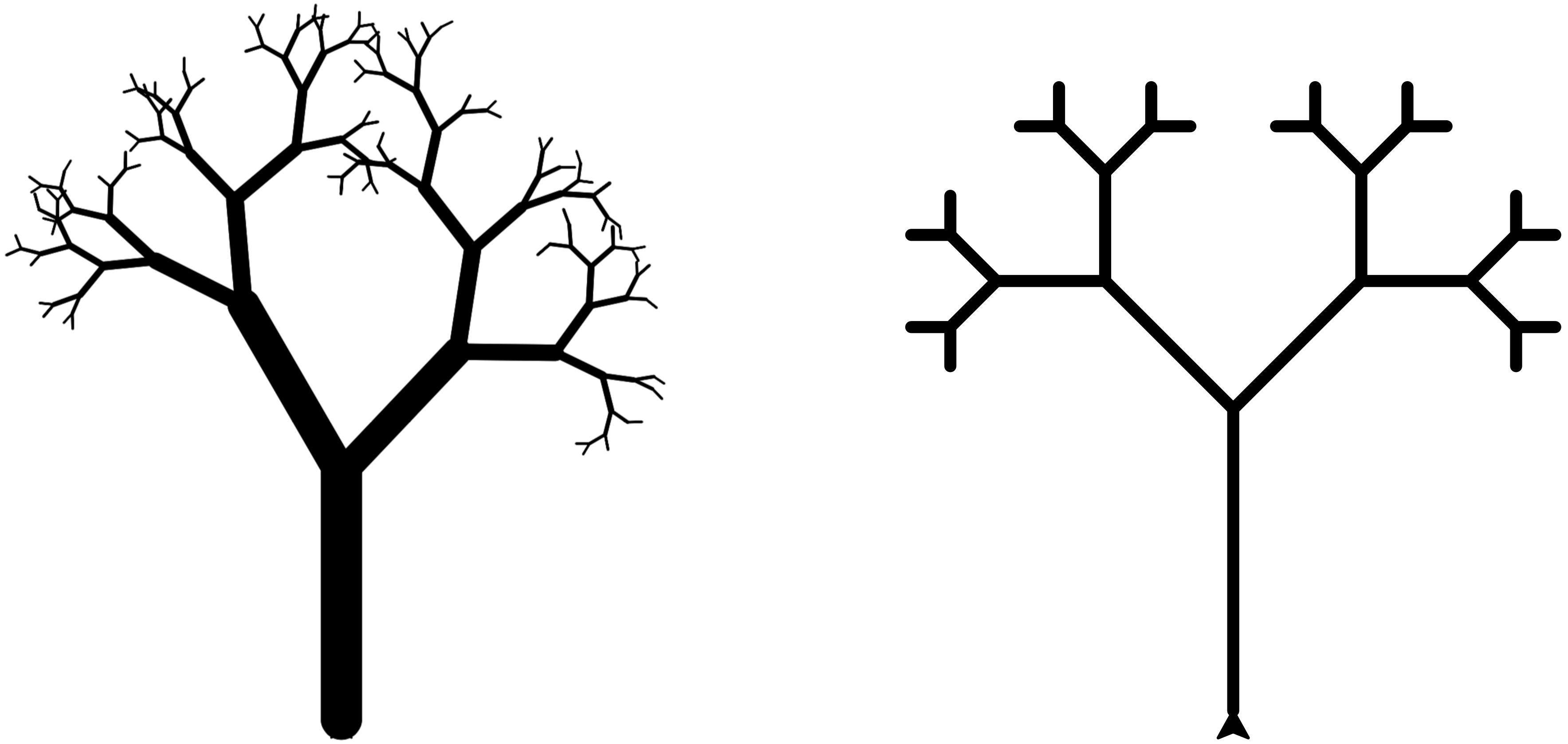
- Why do we care about **invariance**?
    - Though not always necessary for correctness, it is a good property to maintain in recursive functions
    - Our graphical functions will not always work properly if it they are not invariant



Recursive Trees

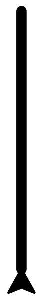
Recursive Trees

- Let's draw some trees using turtle graphics and recursion

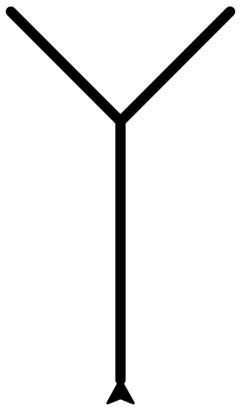


One more recursive example: Trees

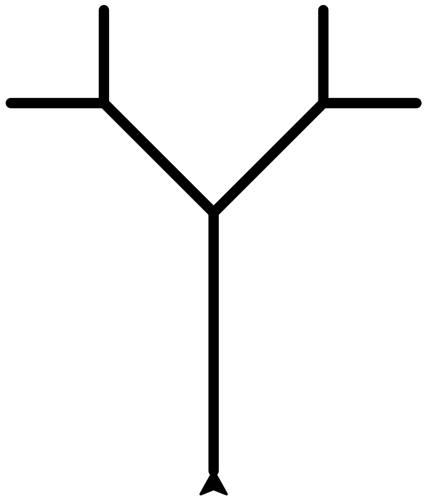
- Example: Draw recursive trees as shown; count and return # branches drawn
  - What is our base case? Recursive case?
  - *Note:* Assume turtle starts facing north



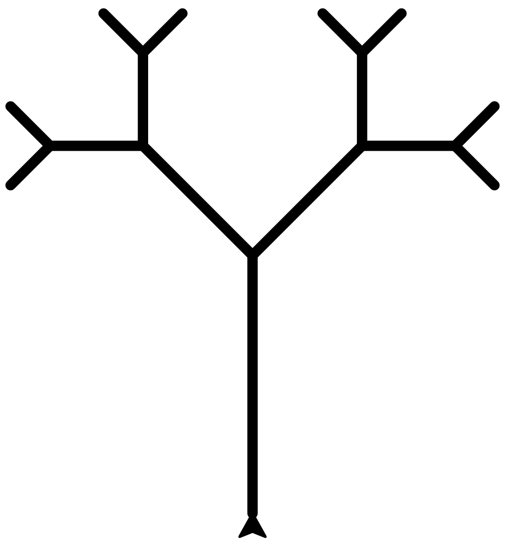
`tree(100, 45, 0.5, 100)`



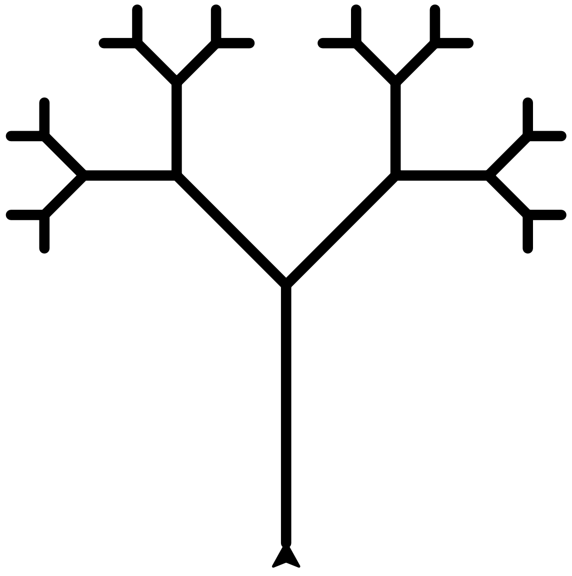
`tree(100, 45, 0.5, 50)`



`tree(100, 45, 0.5, 25)`



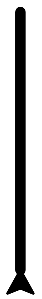
`tree(100, 45, 0.5, 12)`



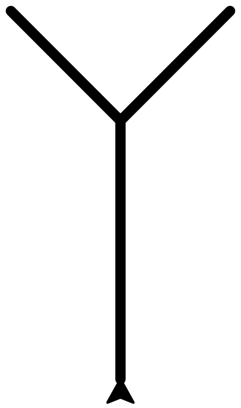
`tree(100, 45, 0.5, 5)`

One more recursive example: Trees

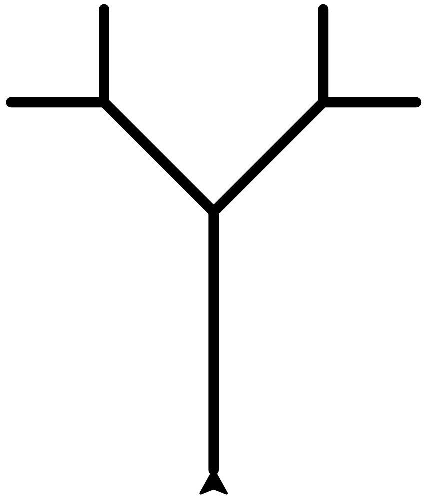
```
def tree(trunk_len, angle, shrink_factor, min_len):
  # trunk_len length of the main (vertical) trunk
  # angle branching angle (angle between a trunk and its
    right or left branch)
  # shrink_factor factor by which each subsequent branch
  shrinks by
  # min_len minimum branch length in our tree
```



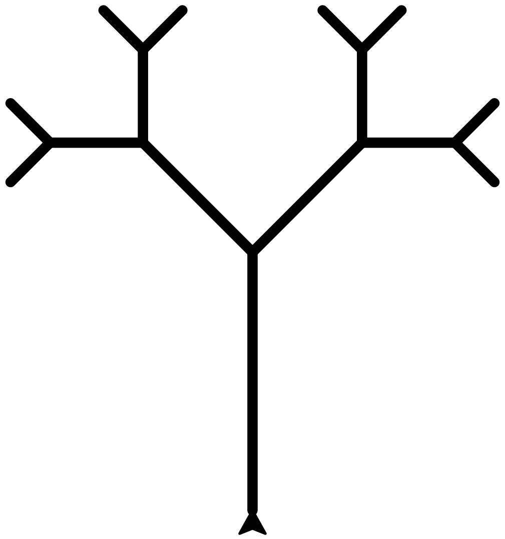
tree(100, 45, 0.5, 100)



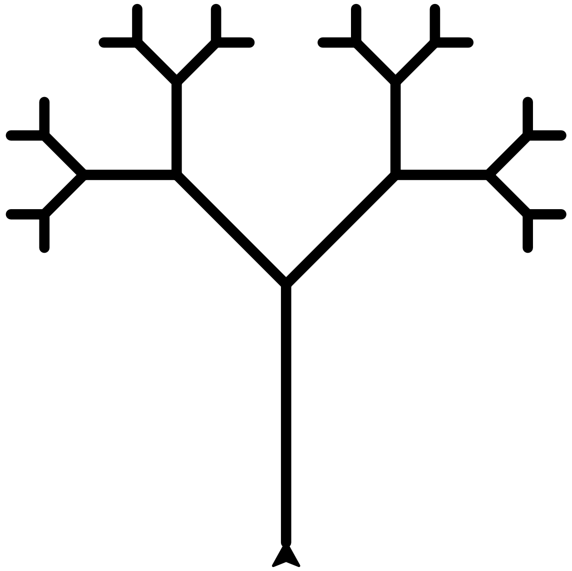
tree(100, 45, 0.5, 50)



tree(100, 45, 0.5, 25)

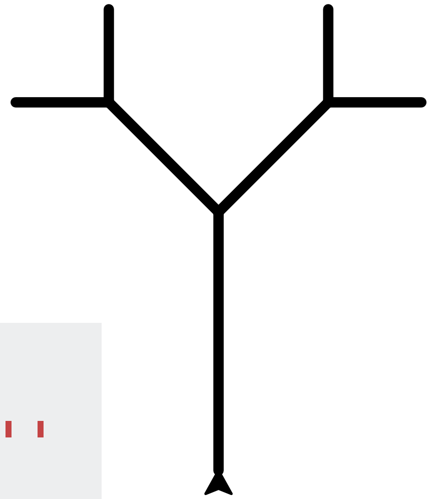


tree(100, 45, 0.5, 12)



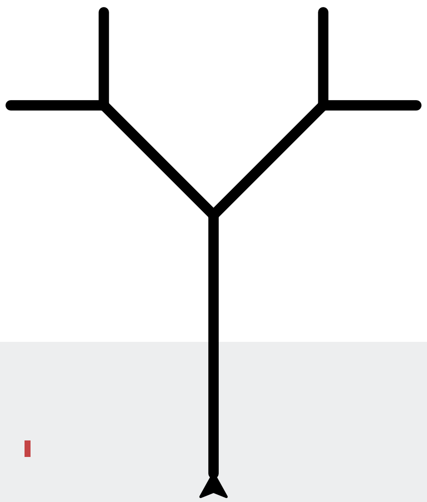
tree(100, 45, 0.5, 5)

Tree: Outline



```
def tree(trunk_len, angle, shrink_factor, min_len):
    '''Draw tree and return number of branches drawn including trunk'''
    # Base case: trunk_len < min_len
        # return 0, don't draw anything!
    # Recursive case:
        # Draw trunk



        # Position for Right branch: Turn right angle
        # Right branch -> shrink trunk, pass along other variables



        # Position for Left branch: Turn left angle*2
        # Left branch -> shrink trunk, pass along other variables



        # Maintain invariance
        # Turn right, then back up to starting position

        # return 1 (for the trunk we drew), plus the sum of the branches
```

Tree: Outline



```python
def tree(trunk_len, angle, shrink_factor, min_len):
    '''Draw tree and return number of branches drawn including trunk'''
    if (trunk_len < min_len): # Base case
        return 0
    else:
        # Draw trunk
        fd(trunk_len)

        # Right branch
        rt(angle)
        right_branch = tree(trunk_len*shrink_factor, angle, shrink_factor, min_len)

        # Left branch
        lt(angle*2)
        left_branch = tree(trunk_len*shrink_factor, angle, shrink_factor, min_len)

        # Maintain invariance
        rt(angle); bk(trunk_len)

        return 1 + right_branch + left_branch
```

Recursion: Wrap Up

What's The Big Deal With Recursion?

- Why choose recursion over iteration?
  - Some problems have a ***natural recursive structure***
  - Using recursion on them leads to elegant and concise solutions
  - Fewer lines of code often correlates with less debugging!
- We will use recursion to search and sort in a few weeks
- Recursion also helps us build and maintain complex data structures
- Downsides: Recursive approaches often have more computational overhead
  - Steeper learning curve (but can be very rewarding once you get the hang of it)
  - To understand recursion you must understand recursion…