Lecture 3

Homework #3: 1.4.1, 1.4.2 a & b, 1.5.1, 1.5.3, 1.5.6, 1.5.7, 1.5.8, Prove that the set of all real numbers is uncountable.

Note that this lecture will likely run over – but the next one is very short, putting us exactly on track again.

Today: Review of proof techniques:

1) Dovetailing
2) Mathematical Induction
3) Pigeonhole Principle
4) Diagonalization

But first some definitions:

Finite Sets:

size = number of elements;
called **cardinality**;
denoted $|A|$ for the set A

Definition: 2 sets A and B are **equinumerous** if there is a bijection $f:A \to B$.

A set is **finite** if it is equinumerous with $\{0,1,2,3, ..., n\}$, $n \in \mathbb{N}$.

$\mathbb{N} = \{0,1,2,3, ..., n\}$.

A set is **countably infinite** if it is equinumerous with $\mathbb{N}$.
{i.e., if there's a way to systematically list the elements of the set}

A set is **countable** if it is finite or countably infinite.

A set of tools for showing a set to be countably infinite:

• Give an explicit bijection between A and some countably infinite set. ($\mathbb{N}$ is most "natural")
• Suggest a way in which it can be enumerated as $\{a_0, a_1, \ldots\}$
• Recall that an (infinite) subset of a countably infinite set is countable.
• Recall that the union of a countable number of countably infinite sets is countable.

Example. Say that A, B, and C are countably infinite sets.
Show that $A \cup B \cup C$ is also countably infinite.

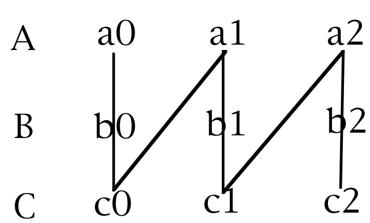
We can write $A = \{a_0, a_1, \ldots\}$
$B = \{b_0, b_1, \ldots\}$
$C = \{c_0, c_1, \ldots\}$

and their union can be listed as $\{a_0, b_0, c_0, a_1, b_1, c_1, \ldots\}$

Again: the intuition is that if you can list the elements, then the set is countable.

But you want to be sure that in your itemization, you're really including everything (and not skipping something). So you need a systematic way of being sure each element is listed:

Dovetailing



An actual bijection $g: \mathbb{N} \rightarrow A \cup B \cup C$ is

$g(n) =$ a_m, $m=(n / 3)$, when $(n \% 3 = 0)$
b_m, $m=(n / 3)$, when $(n \% 3 = 1)$

c_m, m=(n / 3), when (n % 3 = 2)

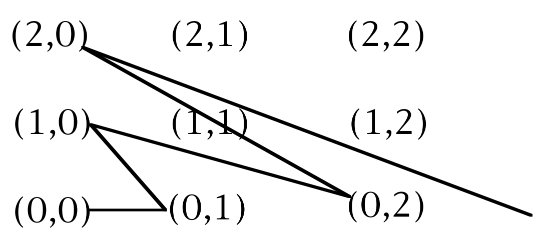
Note: when will dovetailing be a useful tool in this course? toward the *end* of the course.

Another example. (illustrating that the union of a countably infinite collection of countably infinite sets is countable)

Show that $\mathbb{N} \times \mathbb{N}$ is countable.

$\mathbb{N} \times \mathbb{N} = \cup(\{0\}\times \mathbb{N}, \{1\}\times \mathbb{N}, \{2\}\times \mathbb{N}, \ldots)$

can't visit them in quite the same way as last time, so:



if m is the vertical coordinate and n is the horizontal coordinate, then the pair (m,n) is visited kth, where $k = 1/2[(m+n)^2 + 3m+n]$

i.e., $f(m,n)= 1/2[(m+n)^2 +3m+n]$ is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$

Principle of Mathematical Induction

Let A be a set of natural numbers such that

1) $0 \in A$

and 2) for each n, if $\{0,1, \ldots, n\} \in A$, then $(n+1) \in A$.

Then $A = \mathbb{N}$

Used to prove: “For all natural numbers n, P is true.”

Example. Show by induction that

$$0^3 +1^3 +2^3 + \ . \ . \ . + n^3 = (0+1+2+ \ldots +n)^2$$

Basis. Let n = 0; $0^3 = (0)^2 = 0$

I.H. Assume that for some $n \geq 0$,

$0^3 + 1^3 + 2^3 + \ldots + m^3 = (0+1+2+ \ldots +m)^2$, $m \leq n$

Induction Step

Show $0^3 + 1^3 + 2^3 + \ldots + (n+1)^3 = (0+1+2+ \ldots +n+n+1)^2$

$0^3 + 1^3 + 2^3 + \ldots + (n+1)^3$

$= (0+1+2+ \ldots +n)^2 + (n+1)^3$

$= ((n)(n+1)/2)^2 + (n+1)^2(n+1)$

$= ((n^2 +n)/2)^2 + n(n+1)^2 + (n+1)^2$

$= (n^4 + 2n^3 + n^2)/4 + 4/4(n)(n+1)^2 + 4/4(n+1)^2$

$= (n^4 + 6n^3 + 13n^2 + 12n + 4)/4$

$= (n^2 + 3n + 2)^2/4$

$= ((n+1)(n+2)/2)^2$

$= (0 + 1 + 2 + \ldots + n + n+1)^2$

Note the types of problems on which induction might be used:

1) proofs of correctness and time complexity.

2) induction on length of the string; induction on the length of a computation.

The Pigeonhole Principle

If A and B are finite sets and $|A| > |B|$,
then there is no 1-1 function from A to B.

[try to pair off elements of A with elements of B;
"pigeons" and "pigeonholes"; but you have an extra pigeon.]

Note that we won't use this too often - but it is occassionally useful. (for instance for the intuition of why there must be cycles in derivations of certain lengths for regular grammars - Pumping Lemma.)

The Diagonalization Principle

The intuition is as follows: we're going to systematically organize/itemize/list things. then we'll show that there's a way of creating a legitimate item that should be itemized, but that doesn't actually fit anywhere in the listing.

The principle: let R be a binary relation on a set A, and let D (the "diagonal" set for R) be

$$\{a: a \in A \text{ and } (a,a) \notin R\}$$

For each $a \in A$, let $Ra = (b: b \in A \text{ and } (a,b) \in R\}$

Then D is distinct from each Ra.

Look at it pictorially:

> Let A = {a, b, c, d} and let the following table give a relation R: A × A.

	a	b	c	d	
a	x		x		Ra
b		x			Rb
c			x		Rc
d	x				Rd

Now look at the diagonal: x, x, x, _
and flip it: _, _, _, x

This is not the same as any other row.
The same idea can be used for infinite sets.

Example. Show that [0,1) is uncountable.

Assume the contrary, i.e., that [0,1) is countable. Then the elements can be itemized as follows:

$x_1 = 0.\ a_{11}\ \ a_{12}\ \ \ a_{13}\ \ \ ...$
$x_2 = 0.\ a_{21}\ \ a_{22}\ \ \ a_{23}\ \ \ ...$
$x_3 = 0.\ a_{31}\ \ a_{32}\ \ \ a_{33}\ \ \ ...$
etc.

Where each x_i is the decimal expansion of a number between 0 and 1.

Now corrupt each digit along the diagonal - i.e.,
let d_{11} be some $n \in \{1, .., 8\} \neq a_{11}$
let d_{22} be some $n \in \{1, ..., 8\}$ a $\neq$ $_{22}$
etc.

but then 0. $d_{11}d_{22}$... isn't enumerated!